RMO-2009 Solutions by Dr. V. V. Acharya

2. Show that there is no integer a such that $a^2 - 3a - 19$ is divisible by 289
Solution: Suppose there exists a such that

$289|a^2-3a-19 \Rightarrow 289|4(a^2-3a-19)$ i.e. $4a^2-12a+9+85 = 289k$

Now $17|289, 17|85 \Rightarrow 17|(2a-3)^2 \Rightarrow 17|2a-3$
$\Rightarrow 17^2|(2a-3)^2$ and $17^2|289 \Rightarrow 17^2|85$, a contradiction

3. Show that $3^{2008} + 4^{2009}$ can be written as product of two positive integers each of which is larger than 2009^{182}.
Solution: We use Sophie Germain's identity

$$\begin{aligned}
a^4 + 4b^4 &= (a^2+2b^2)^2 - (2ab)^2 \\
&= (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2) \\
\text{Now} \quad & 3^{2008} + 4(4^{2008}) \\
&= (3^{1004} - 2(3^{502})4^{502} + 4^{1004} + 4^{1004}) \\
& \times (3^{1004} + 2(3^{502})4^{502} + 4^{1004} + 4^{1004}) \\
&= ((3^{502} - 4^{502})^2 + 4^{1004})((3^{502} + 4^{502})^2 + 4^{1004}).
\end{aligned}$$

Note that each factor is greater than 4^{1004}. It suffices to show that $4^{1004} > 2009^{182}$ i.e.
Now, $4^{1004} = 2^{2008} = (2^{11})^{182} \cdot 2^6 > (2048)^{182} > 2009^{182}$.
Hence, each factor is greater than 2009^{182}.