MATHEMATICAL OLYMPIAD CAMP-SHILLONG-OCT.2003

Some Problems In Algebra/Compiled by MBR.

§2.1 ALGEBRAIC INEQUALITIES (Part 1)

P2.101. Which of the numbers 31^{17} and 17^{22} is greater?

P2.102. Among the numbers, $31^{17}, 17^{22}$ and 4^{43}, which is the greatest and which is the smallest?

P2.103. (i)Which of the numbers e^{π} and π^{e} is greater?

(ii)Which of the numbers $2.8^{2.9}$ and $2.9^{2.8}$ is greater?

(iii) Which of the numbers $2.8^{2.83}$ and $2.83^{2.8}$ is greater?

P2.104. Consider the fractions

$$\frac{4567890123}{7890123456} \quad \text{and} \quad \frac{4567890124}{7890123458}$$

Determine which is greater by giving an algebraic argument; do not multiply the numbers involved.

P2.105 .Which of the following fractions is greater? Answer with justification.

$$\frac{10^{99}+1}{10^{98}+1} \quad \text{and} \quad \frac{10^{100}+1}{10^{99}+1}$$

P2.106. Which of the following fractions is greater? Answer with justification.

$$\frac{10^{99}-10^{98}+1}{10^{98}-10^{97}+1} \quad \text{and} \quad \frac{10^{100}-10^{99}+1}{10^{99}-10^{98}+1}$$

P2.107. Prove that

$$\left(\frac{2}{1}.\frac{4}{3}\cdots\frac{1328}{1327}\right)^2 > 1993$$

P2.108. Prove that

$$\left(\frac{2}{1}.\frac{4}{3}\cdots\frac{100}{99}\right) > 12$$

P2.109. Prove that if $x_1, x_2, \ldots, x_n$ are n positive real numbers whose sum is 1, then

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}$$

P2.110. Let a, b be positive real numbers. Prove that

$$\Big[(a+b)(a^2+b^2)(a^3+b^3)\cdots(a^n+b^n)\Big]^2 > (a^{n+1}+b^{n+1})^n.$$

P2.111. Prove that for each positive integer n bigger than 1 we have

$$2^n > 1 + n.2^{(n-1)/2}$$

P2.112. Prove that for each positive integer n we have

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

P2.113. Prove that for each positive integer n the sum

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots\frac{1}{3n+1}$$

is greater than 1.

P2.114. Which real numbers, if any, satisfy the inequalities

$$\frac{1}{x}yz < -x+y+z \qquad (I)$$

$$\frac{1}{y}zx < -y+z+x \qquad (II)$$

$$\frac{1}{z}xy < -z+x+y \qquad (III)$$

P2.115. Suppose that the positive real numbers x, y, z satisfy $x^3y^2z^4 = 7$, prove that

$$2x+5y+3z \geq 9(\frac{525}{128})^{\frac{1}{9}}$$

P2.116.(A Canadian IMO Question.)

Let a, b, c be positive real numbers satisfying the condition $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

P2.117. If a, b, x, y are real numbers, then

$$\left(a^2+b^2\right)\left(x^2+y^2\right) \geq \left(ax+by\right)^2$$

P2.118(i) Let a, b be positive numbers satisfying $a + b = 1$. Show that

$$(a+\frac{1}{a})^2+(b+\frac{1}{b})^2 \geq \frac{25}{2}. \qquad (*)$$

(ii) Let a, b, c be positive real numbers satisfying the condition $a + b + c = 1$. Prove that we have

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2 \geq \frac{100}{3}$$

P2.119. Let $\{a_i\}_{i=1,\ldots n}$ be positive real numbers satisfying the condition

$$\sum_{i=1}^{n} a_i = 1,$$

then

$$\left(\sum_{i=1}^{n}\left(a_i+\frac{1}{a_i}\right)^2\right) \geq \frac{(n^2+1)^2}{n}$$

P2.120. An IMO problem. Find all real numbers a for which there are $x_k \geq 0, k = 1, 2 \cdots 5$ such that

$$\sum_{k=1}^{5} k.x_k = a \qquad (I)$$

$$\sum_{k=1}^{5} k^3.x_k = a^2 \qquad (II)$$

$$\sum_{k=1}^{5} k^5.x_k = a^3. \qquad (III)$$

P2.121. Which number in the sequence

$$2^{1/2}, 3^{1/3}, 4^{1/4}, \ldots, n^{1/n}, \ldots$$

is the largest ? Answer with a brief justification.

P2.122. If x, y are positive real numbers, show that

$$x^y + y^x > 1$$

P2.123. Prove that if x, y are non-negative real numbers we have the inequality

$$(x^3 + y^3)(x + y) \geq (x^2 + y^2)^2$$

P2.124. Let x and y be real numbers such that $x^3 + y^3 > 1$. Prove that $x^2 + y^2 > 1$.

P2.125.Prove that for each positive integer n, we have

$$\frac{1}{n+1}\left[1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right] \geq \frac{1}{n}\left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right]$$

§2.2 ALGEBRAIC INEQUALITIES (Part 2).

P2.201. Verify the following different forms of a fundamental algebraic inequality: If a, b are real numbers, then

$$i)[a-b]^2 \geq 0$$

$$ii)a^2 - 2ab + b^2 \geq 0$$

$$iii)a^2 - ab + b^2 \geq ab;$$

$$iv)a^2 + b^2 \geq 2ab;$$

$$v)[a+b]^2 \geq 4ab.$$

$$vi)2[a^2 + b^2] \geq [a+b]^2$$

and

$$vii)2[a^2 + b^2] \geq [a+b]^2 \geq 4ab$$

P2.202. Prove that if a, b are real numbers with

$a + b$ positive, then

$$i) a^3 + b^3 \geq ab[a + b],$$

$$ii) 4[a^3 + b^3] \geq [a + b]^3$$

$$iii) [a^3 + b^3]/2 \geq [(a + b)/2]^3$$

P2.203. Give examples to show that the inequalities in i)-iii) in Problem P2.202 are not valid for all real numbers.

P2.204. Prove that if a, b, c are positive real numbers

$$6abc \leq a^2(b + c) + b^2(c + a) + c^2(b + a) \leq 2(a^3 + b^3 + c^3)$$

P2.205. Prove that if a, b, c are positive real numbers

$$\frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} + \frac{a^2 + c^2}{a + c} \geq a + b + c$$

.

P2.206. Verify the "A.M.-G.M." inequality for two positive real numbers a, b; namely,

$[a + b]/2 \geq \sqrt{ab}$.

P2.207. Verify the following natural generalisation of the above result : if $a_1, \ldots, a_n$ are positive numbers, then

$A.M.(a_1, \ldots a_n) := [a_1 + \ldots + a_n]/n$

$\geq G.M.(a_1, \ldots, a_n) := (a_1 \ldots a_n)^{1/n}$

Remark. Note that in $\sum$ and $\prod$ notation , the inequality in P2.207 takes the form

$$(\sum_{i=1}^{n} a_i)/n \geq (\prod_{i=1}^{n} a_i)^{1/n}$$

. P2.208. Prove that if a, b, c, d are positive real number s,then

$$a/b + b/c + c/d + d/a \geq 4$$

P2.209. Let $b_1, \ldots b_n$ be a permutation of the n positive numbers $a_1, \ldots a_n$. Then we have

$$a_1/b_1 + a_2/b_2 + \ldots + a_n/b_n \geq n$$

P2.210. Let a, b, c be positive. Then we have

$$(a + b)(b + c)(c + a) \geq 8abc$$

with strict inequality holding if a, b, c are not all equal.

P2.211. Let a, b, c be positive. Suppose we have

$$(1 + a)(1 + b)(1 + c) = 8.$$

Then $abc \leq 1$.

P2.212. Let a, b, c, d be positive real numbers satisfying $abcd = 1$.

Prove that

$$(1 + a)(1 + b)(1 + c)(1 + d) \geq 16$$

P2.213. Prove that

$$(n!)^{2/n} < (n + 1)(2n + 1)/6$$

for each integer $n \geq 2$.

P2.214. Prove that

$$(n!)^{3/n} < n(n + 1)^2/4$$

for each integer $n \geq 2$.

P2.215. Let a, b, c be positive real numbers all smaller than 1 satisfying the condition that $a + b + c = 2$. Prove that

$$[a/(1 - a)][(b/(1 - b)][c/(1 - c)] \geq 8. \qquad (*)$$

P2.216. Do there exista, b, c satisfying the conditions in P2.215 for which we have equality in (*)?

P2.217. Let a, b, c be positive real numbers satisfying $a+b+c=1.$. Prove that

$$(1+a)(1+b)(1+c) \geq 8(1-a)(1-b)(1-c). \qquad (*)$$

P2.218. Do there exist a, b, c satisfying the conditions in P2.217 for which we have equality in (*)?

P2.219. Let a, b, c, d be real numbers such that $a+b+c+d=1$.

i) We then have

$$ab+bc+cd+da \leq 1/4.$$

ii) (INMO-1993) If, further, a, b, c, d are all non-negative, then

$$ab+bc+cd \leq 1/4.$$

iii) If a, b, c, d are real numbers satisfying; then $a+b+c+d=1$ then $ab+bc+cd=a^2$ can be made arbitrarily large. In particular, ii) is not valid for arbitrary real numbers a, b, c, d.

P2.220.Let $a_1, a_2, a_3, \ldots a_n$ be real numbers.Then

$$i) a_1a_2+a_2a_3+\cdots+a_na_1 \leq a_1^2+\cdots+a_n^2;$$

$ii)$ if

$$\sum_{i=1}^{n} a_i = 1,$$

with each a_i non-negative, then

$$a_1a_2+a_2a_3+\cdots+a_na_1 \leq 1/4$$

P2.221.For real numbers a, b, c, d let

$$A := ab + ac + ad + bc + bd + cd.$$

Suppose that a, b, c, d satisfy $a^2 + b^2 + c^2 + d^2 = 1$.Prove that $A \geq -1/2$.

P2.222. Show that there exist quadruples a, b, c, d of real numbers satisfying the condition $a^2 + b^2 + c^2 + d^2 = 1$ for which the quantity A in P2.221 attains the value $-1/2$.

P2.223. Let real numbers a, b, c, d satisfy $a^2 + \cdots + d^2 = 1$ and $a+b+c+d = 0$.

Then

$$-1 \leq ab + bc + cd + da \leq 0. \qquad (I)$$

Further, the numbers

$$ab + cd, ac + bd, ad + bc \in [-1/2, 1/2]. \qquad (II)$$

P2.224. Let a, b, c, d be nonnegative real numbers satisfying the condition $a+b+c+d = 1$. Find the maximum value of each of the following expressions:

$$(i) ab + cd; (ii) ab + cd + ac; (iii) ab + ac + ad + bd.$$

P2.225.If a, b are positive real numbers, then

$$(a + b)(1/a + 1/b) \geq 4. \qquad (*)$$

P2.226. Let $a_1, a_2, a_3, \ldots a_n$ be n real numbers all greater than 1 and such that $|a_k - a_{k+1}| \leq 1$ for $1 \leq k \leq n-1$. Show that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1} \leq 2n - 1$$

.

P2.227. If $n > 1$, then

$$2!4!6! \cdots (2n)! > (n+1)!^n$$

P2.228.a, b, c, x, y, z, k are non-negative real numbers such that

$$a + x = b + y = c + z = k.$$

Prove that

$$ay + bz + cx \leq k^2$$

P2.229. If $a > b > 0$ show that

$$\frac{1 + b + b^2 + b^3 + \ldots b^9}{1 + b + b^2 + b^3 + \ldots b^{10}}$$
$$\geq \frac{1 + a + a^2 + a^3 + \ldots + a^9}{1 + a + a^2 + \ldots + a^{10}}$$

P2.230. Let x be a real number, and let $a = x^5 - x^3 + x$. Prove that $x^6 \geq 2a - 1$. (INMO 94)

P2.231. Let x, y be positive real numbers such that $x + y = 2$. Show that $x^3y^3(x^3 + y^3) \leq 2$.

P2.232.Let $a_1 \leq a_2 \leq \ldots \leq a_n$ be n real numbers such that

$$\sum_{j=1}^{n} a_j = 0$$

Let $E(= E(a_1, a_2, a_3, \ldots, a_n))$ denote the expression $na_1a_n + \sum_{j=1}^{n}(a_j)^2$. Show that $E \leq 0$.

P2.233. Let x, y be non - negative real numbers such that

$$2x + y + \sqrt{2xy + 3x^2 + y^2} = 5 \qquad (*)$$

Prove that$x^2y < 1$

P2.234. Prove that for positive real numbers a, b, c we have

$$a^a b^b c^c \geq (abc)^{\left[\frac{a+b+c}{3}\right]}$$

P2.235. Let real numbers a, b, c satisfy the condition:
$|at^2 + bt + c| \geq 1$ for each t satisfying

$$0 \leq t \leq 1 \cdots \qquad (A)$$

Part I: Prove that

$$|a| + |b| + |c| \geq 17 \cdots \qquad (B)$$

Part II: Prove further that equality can hold in (B) by constructing an example

P2.236. Let a, b, c, d be positive real numbers. Show that

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \geq \frac{1}{abc} + \frac{1}{bcd} + \frac{1}{cda} + \frac{1}{dab}.$$

P2.237. Let $a_1, a_2, \ldots, a_n$ be real numbers such that each $a_i \geq 1$. Show that

$$2^{n-1}(1 + a_1a_2a_3 \cdots a_n) \geq (1 + a_1)(1 + a_2)(1 + a_3) \cdots (1 + a_n).$$

§2.3 **Quadratic expressions and equations**

P2.301.Let a, b, c, d be real numbers satisfying the condition

$$b + d = 2ac.$$

Prove that at least one of

the equations

$$x^2 + 2ax + b = 0$$

and

$$x^2 + 2cx + d = 0$$

has real roots.

P2.302.Show that there are infinitely many pairs (a, b)

of relatively prime (not necessarily positive) integers

such that the quadratic equations

$$x^2 + ax + b = 0$$

and

$$x^2 + 2ax + b = 0$$

both have integer roots.

P2.303.Let a, b, c be integers with $b \neq -1$. If c is a root of $x^2 + ax + b + 1$, show that the integer $a^2 + b^2$ is composite.

P2.304.Let q be an integer.Show that the quadratic equation

$$x^2 + 7x - 14(q^2 + 1) = 0$$

has no integer root (CRMO - 1995)

P2.305.Find all integer values of a for which the quadratic expression

$$(x + a)(x + 1991) + 1$$

can be factored as

$$(x + b)(x + c),$$

for some integers b, c (CRMO - 1991)

P2.306. If a, b, c are odd integers, prove that that the roots of the quadratic equation

$$ax^2 + bx + c = 0$$

cannot be rational numbers.

P2.307. Find the number of quadratic polynomials
$ax^2 + bx + c$, which satisfy the following conditions

(i) a, b, c are distinct

(ii) $a, b, c \in \{1, 2, 3, \cdots 1999\}$ and (iii)$x + 1$ divides $ax^2 + bx + c$ (CRMO - 1999)

P2.308 Suppose a, b, c are three real number such that the quadratic equation

$$x^2 - (a + b + c)x + (ab + bc + ca) = 0$$

has roots of the form $\alpha + i\beta (\in C)$ where $\alpha > 0$ and $\beta \neq 0$ are real numbers show that

(1) the numbers a, b, c are positive;

(2) the numbers $\sqrt{a}, \sqrt{b}, \sqrt{c}$ form the sides of a triangle.(INMO - 1998)

P2.309 Given any four distinct positive real numbers,show that one can always choose three among them, say A, B, C such that either all the three equations $Ax^2 + x + C = 0, Bx^2 + x + C$ and $Cx^2 + x + A$ have real roots or all the three equations have non-real roots.

P2.310 Let $p(x)$ and $q(x)$ be two quadratic polynomials with integer coefficients. Suppose they have a (complex) non-rational zero in common.

Show that for some rational number $r, p(x) = rq(x)$ holds.

(There is no problem P2.311.)

P2.312. Consider the quadratic equations $x^2 - px + q = 0$ and $x^2 - p'x + q' = 0$ (over complex numbers). Suppose we are given that $|p - p'| < 1/1000$ and $|q - q'| < 1/1000$. Can it happen that the greater root of the first equation differs from the greater root of the second equation by more than 1000 in absolute value?

§2.4 **Cubics and polynomials of higher degree; rational functions.**

P2.401. Let $f(x) = x^3 - 6x^2 + 16x - 8$pp. Suppose that a and b are real numbers such that $f(a) = 9$ and $f(b) = 7$. Find $a + b$.

P2.402. Let α and β be two real numbers such that $\alpha^3 - 3\alpha^2 + 5\alpha - 17 = 0$ and $\beta^3 - 3\beta^2 + 5\beta + 11 = 0$. Find $\alpha + \beta$.

P2.403.Let $f(x)$ and $g(x)$ be polynomials of degree 6 with real coefficients.

Suppose that $f(x) = a_0 + a_1x + a_5x^5 + a_6x^6$ and $g(x) = b_0 + b_1x + b_2x^2 + b_6x^6$.

Let $d(x)$ be the g.c.d. of $f(x)$ and $g(x)$. (a) What is the largest possible degree that $d(x)$ can have?

(b) If $b_2 \neq 0$, show that $deg(d(x)) \leq 4$.

(c) Show by an example that the degree 4 is actually attained (under the $b_2 \neq 0$ condition.

P2.404.Determine the largest real number A satisfying the condition that there exist real numbers B, C, D, E such that for each real number t satisfying $|\, t \,| \leq 1$ we have

$$0 \leq At^4 + Bt^3 + Ct^2 + Dt + E \leq 1$$

P2.405. Suppose a and b are two positive real numbers such that the roots of the cubic equation $x^3 - ax + b = 0$ are all real. If α is a root of this equation with minimal absolute value prove that

$$\frac{b}{a} < \alpha \leq \frac{3b}{2a}.$$

P2.406. Find the set of all pairs (n, r), with n a positive integer, r a real number, for which the polynomial $(x+1)^n - r$ is divisible by $2x^2 + 2x + 1$. (47th Polish Mathematical Olympiad(1995-96),March '96)

(Ans.: The required set is given by

$$\{(4m, (-4)^{-m}) \mid m \in \mathbf{N}\})$$

P2.407. Without attempting to solve the equation, show that the equation

$x^4 + 1 = x^3$ has no real solution(NERMO, Nov.'89).

P2.408. Determine all real numbers (if any) satisfying the following two equations simultaneously:

$(I)x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$; $(II)x^4 + 3x^3 + 3x^2 + x = 0$. (NERMO,Sept. 94)

P2.409.Let $f(x) = (1 + x + x^2 + x^3)^{25}$

$= a_0 + a_1x + a_2x^2 + \cdots + a_{75}x^{75} =$

$$\sum_{i=0}^{75} a_i x^i$$

Suppose that $A = a_0 + a_2 + \ldots + a_{74} =$

$$\sum_{i=0}^{37} a_{2i}$$

Prove that 2^{10} divides A. (Hint: Compute $f(1), f(-1)$)

P2.410.Find an integer a such that $x^2 - x + a$ is a factor of $x^{13} + x + 90$. (Ans.: $a = 2$)

P2.411.Determine the remainder when $(x+1)^n$ is divided by $(x-1)^3$.

P2.412.Prove that there exist integers a and b such that the polynomials

$$x^{13} - 233x - 144$$

and

$$x^{15} + ax + b$$

have a common non-trivial factor. (Hint: Think Fibonacci.)

P2.413. Solve

$$(x+1)(x+3)(x+5)(x+7)(x+9)(x+11) + 225 = 0$$

P2.414. If α, β, γ are the roots of $x^3 - x - 1$, compute

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}$$

P2.415. Determine all real a such that

$$a^n + a^{-n}$$

is an integer for any integer n. (10th Nordic Mathematical Contest.)

P2.416. If

$$a + 1/a = -1$$

find

$$a^{99} + 1/a^{99}.$$

P2.417. If a, b, c are real numbers satisfying the equation $9a+11b+29c=0$, prove that the equation

$$ax^3 + bx + c = 0$$

has a solution in $[0, 2]$.

P2.418.Solve for real numbers a and b:

$$ab^2 = 15a^2 + 7ab + 15b^2;$$

$$a^2b = 20a^2 + 3b^2$$

P2.419 Solve the following system of equations:

$$(a-2)(b-2) = 4$$

$$(b-3)(c-3) = 9$$

$$(c-4)(a-4) = 16$$

P2.420. Solve the following system of equations for real numbers a, b, c, d, e.

$$3a = (b+c+d)^3, 3b = (c+d+e)^3, 3c = (d+e+a)^3,$$

$$3d = (e+a+b)^3, 3e = (a+b+c)^3$$

. P2.421. Suppose that the polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ satisfies $f(1) = 10, f(2) = 20$ and $f(3) = 30$. Determine the value of $f(12) + f(-8)$.

P2.422. Let $a_1, a_2, \ldots a_n$ be distinct integers. Prove that the polynomial $h(x) = \prod_{i=1}^{n}(x - a_i)^2 + 1$ is irreducible over the integers.

P2.423. Prove that $x^5 + 2x + 1$ is irreducible over the integers.

P2.424. Show that $x^8 + 98x^4 + 1$ is reducible over the integers.

P2.425. Let a, b, c be three real numbers satisfying $1 \geq a \geq b \geq c \geq 0$. Prove that if λ is a (real or (complex) non-real) root of the cubic equation $x^3 + ax^2 + bx + c = 0$, then $|\lambda| \leq 1$.

P2.426. Consider the cubic equation $x^3 - 3px^2 + 3q^2x - r^3 = 0$. If p, q, r are roots of this equation, show that $p = q = r$.

§2.5 **Functions and functional equations.**

P2.501. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a function such that

$$f(x) \leq x \quad \forall x \in \mathbf{R}$$

and

$$f(x+y) \leq f(x) + f(y) \quad \forall x, y \in \mathbf{R}$$

Prove that $f(x) = x \quad \forall x \in \mathbf{R}$.

P2.502.

Find all functions defined on the set $\mathbf{R}^+$ of all positive real numbers which take positive real values and satisfy the conditions

$$f(xf(y)) = yf(x) \quad \forall x, y \in \mathbf{R}^+ \qquad (1)$$

and

$$f(x) \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty \qquad (2)$$

P2.503. Determine all functions $f : \mathbf{N} \cup \{0\} \longrightarrow \mathbf{N} \cup \{0\}$ satisfying the condition

$$f(m + f(n)) = f(f(m)) + f(n) \qquad \forall m, n \in \mathbf{N} \cup \{0\}$$

(37th IMO, Mumbai, July 1996)

P2.504. Let $\mathbf{R}$ denote the set of all real numbers. Does there exist a function $f : \mathbf{R} \longrightarrow \mathbf{R}$ which satisfies the following conditions:

a) there exists a positive integer M such that

$$-M \leq f(t) \leq M \qquad \forall t \in \mathbf{R};$$

b) $f(1) = 1$;

c) we have

$$f(t + \frac{1}{t^2}) = f(t) + (f(\frac{1}{t}))^2$$

.

(Switzerland M.O.)

P2.505.Determine all functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that

$$f(x^2 + f(y)) = f(x)^2 + y \qquad \forall x, y \in \mathbf{R}$$

(Q.2, 33rd IMO, 1992, Moscow, Russia)

P2.506.Show that there exists a function $f : \mathbf{Q}^+ \longrightarrow \mathbf{Q}^+$ such that

$$f(xf(y)) = f(x)/y \qquad \forall x, y \in \mathbf{Q}^+.$$

(Q.4, 31st IMO, 1990, Beijing, PRC)

P2.507. Let S be the set of all real numbers strictly greater than -1. Find all functions $f : S \longrightarrow S$ satisfying the two conditions:

$$i) f(x + f(y) + xf(y)) = y + f(x) + yf(x) \qquad \forall x, y \in S;$$

$ii) f(x)/x$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x < \infty$.

P2.508. Let f be a function from the set $\mathbf{N}$ (of natural numbers) to $\mathbf{N}$.

Suppose that $f(n+1) > f(n)$ and $f(f(n)) = 3n$ for all $n \in \mathbf{N}$ Determine $f(1992)$.

P2.509(IMOTC 1995)

Denote the set of all subsets of a set X by $P(X)$ and let E(X) be the set of all subsets of X with even number of elements. Suppose $f : E(X) \to \mathbb{R}$ is a map satisfying the conditions

(i) there exists $D_0 \in E(X)$,such that $f(D_0) > 1995$

(ii) if B, C are disjoint members of $E(X)$ then $f(B \cup C) = f(B) + f(C) - 1995$

Show that these exist a subset P of X such that (i) P is non - empty and for each $D \in E(P), f(D) > 1995$

(ii) for each $D \in E(X \setminus P), f(D) \leq 1995$

P2.510(IMOTC 2000)

Suppose $f : \mathbf{Q} \longrightarrow \{0, 1\}$ is a function with the property that for $x, y \in \mathbf{Q}$, the condition $f(x) = f(y)$ implies $f(x) = f(\frac{x+y}{2}) = f(y)$ If $f(0) = 0$ and $f(1) = 1$, show that $f(q) = 1, \forall q \in \mathbf{Q}, q \geq 1$

§2.6 **Miscellaneous Problems in Algebra**

P2.601 If

$$\frac{1}{a^2-bc}+\frac{1}{b^2-ca}+\frac{1}{a^2-ab}=0$$

prove that :

$$\frac{a}{(a^2-bc)^2}+\frac{b}{(b^2-ca)^2}+\frac{c}{(a^2-ab)^2}=0$$

P2.602. If

$$\frac{a}{a^2-bc}+\frac{b}{b^2-ca}+\frac{c}{a^2-ab}=0$$

prove that :

$$\frac{a}{(a^2-bc)^2}+\frac{b}{(b^2-ca)^2}+\frac{c}{(a^2-ab)^2}=0$$

P2.603. Let x, y be real numbers satisfying

$$5x(1+\frac{1}{x^2+y^2})=12 \qquad (i)$$

and

$$5y(1-\frac{1}{x^2+y^2})=4 \qquad (ii)$$

Determine them.

P2.604. Show that if m is a positive integer, there exists a positive integer a such that

$$(\sqrt{2}-1)^m=\sqrt{a+1}-\sqrt{a}.$$

P2.605. Show that given a positive real number ϵ , there exist integer m, n, p such that $0 < m\sqrt{2}+n\sqrt{3}+p\sqrt{5} < \epsilon$

P2.606. Let x, y be real numbers satisfying

$$(x+\sqrt{x^2+1})(y+\sqrt{y^2+1})=1$$

Show that $x+y=0$.

P2.607. Find all (real) solutions of

$$\frac{4x^2}{1+4x^2}=y$$

$$\frac{4y^2}{1+4y^2}=z$$

$$\frac{4z^2}{1+4z^2}=x$$

P2.608. Solve the system of equations

$$x+y+xy=19$$

$$y+z+yz=11$$

$$z+x+zx=14$$

P2.609.Show that there are exactly two triplets (x, y, z) of real numbers satisfying

$$x+y^2+z^4=0 \qquad (i)$$

$$y + z^2 + x^4 = 0 \qquad (ii)$$

$$z + x^2 + y^4 = 0 \qquad (iii)$$

P2.610 Suppose a, b, c are distinct real numbers satisfying

$$a + 1/b = b + 1/c = c + 1/a = t$$

Prove that

$$abc + t = 0$$

P2.611 If a, b, c are real numbers satisfying

$$a + 1/b = b + 1/c = c + 1/a,$$

prove that either $a^2b^2c^2 = 1$ or $a = b = c$.

P2.612 Suppose that real numbers $\alpha, \beta, \gamma, \delta$ satisfy the condition

$$\alpha + \beta + \gamma + \delta = \alpha^7 + \beta^7 + \gamma^7 + \delta^7 = 0$$

. Show that

$$\alpha(\alpha + \beta)(\alpha + \gamma)(\alpha + \delta) = 0$$

P2.613.Let A be a subset of the set of $\mathbf{N}$ of all positive integers satisfying the following conditions: 1.$A \neq \emptyset$;

2.For each $n \in A$, $4n$ also belongs to A;

3.For each $n \in A$, the integral part of $\sqrt{n}$ belongs to A,

i.e.,$[\sqrt{n}] \in A$.

Prove that $A = \mathbf{N}$

P2.614.Determine the set of positive integers n such that

$[\frac{n^2}{3}]$ is a prime.

P2.615.Determine the set of positive integers n such that $[\frac{n^2}{5}]$ is a prime.

P2.616.Solve the equation

$$16[x]^2 + 16\{x\}^2 - 24x = 11$$

where $\{x\}$ denotes the fractional part of a real number x, so that $x = [x]+\{x\}$

P2.617. Show that for each positive integer n we have

$$\left[\left(n(n+1)\cdots(n+7)\right)^{1/4}\right] = n^2 + 7n + 6$$

P2.618. Prove that for each positive integer n, the integer $[(2+\sqrt{2})^n]$ is odd.

P2.619. Given any positive integer n show that there exist two distinct positive rational numbers a and b, which are not integers and which are such that

$$a - b, a^2 - b^2, a^3 - b^3, \ldots, a^n - b^n$$

are all integers.

P2.620. Given that m and n are relatively prime integers both greater than one, prove that $log_{10}m/log_{10}n$ is not a rational number.

P2.621. For positive real numbers x and y, define

$x * y = \frac{x+y}{1+xy}$

Compute $2 * 3 * 4 * \ldots 1998$.

P2.622. Prove or disprove the existence of a sequence x_n of positive real numbers such that $\sqrt{x}_{n+2} = \sqrt{x}_{n+1} - \sqrt{x}_n$ for each $n \in \mathbf{N}$