Crux Mathematicorum

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CONTENTS

THE OLYMPIAD CORNER

No. 121

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Another year has passed and with its end we see some changes in the set-up of *Crux*. As of this issue, as well as continuing the Corner, I shall be attempting to assist Bill with some of the work of editing and setting the journal. We will also be joined by a board of associate editors. Laurie Loro has decided that five years of word processing is enough, and with some trepidation we are switching over to LaTeX. My thanks to Laurie for all her efforts.

It is also the occasion to thank those who have contributed problem sets and solutions in the past year. Among these are: *Anonymous, Seung-Jin Bang, Francisco Bellot, O. Bottema, Duane M. Broline, Curtis Cooper, Nicos Diamantis, Mathew Englander, George Evagelopoulos, J. Chris Fisher, Guo-Gang Gao, Douglass L. Grant, the late J.T. Groenman, R.K. Guy, L.J. Hut, Walther Janous, Murray S. Klamkin, Botand Köszegi, Indy Lagu, H.M. Lee, Matt Lehtinen, Andy Liu, U.I. Lydna, John Morvay, Richard Nowakowski, Antonio Leonardo P. Pastor, J. Pataki, Bob Prielipp, Michael Rubenstein, Kevin Santosuosso, Mark Saul, Jonathan Schaer, Toshio Seimiya, M.A. Selby, Robert E. Shafer, Zun Shan, Bruce Shawyer, Shailesh Shirali, D.J. Smeenk, Jordan Tabov, George Tsintsifas, David Vaughan, G.R. Veldkamp, Edward T.H. Wang, and Willie Yong.*

It is perhaps appropriate that the first set of problems of the new year be New Year's problems from China. My thanks to Andy Liu, University of Alberta for translating and forwarding them. They were published in the *Scientific Daily*, Beijing.

1980 CELEBRATION OF CHINESE NEW YEAR CONTEST
February 8, 1980

1. $ABCD$ is a rhombus of side length a. V is a point in space such that the distances from V to AB and CD are both d. Determine in terms of a and d the maximum volume of the pyramid $VABCD$.

2. Let n be a positive integer. Is the greatest integer less than $(3+\sqrt{7})^n$ odd or even?

3. A convex polygon is such that it cannot cover any triangle of area $1/4$. Prove that it can be covered by some triangle of area 1.

4. Denote by a_n the integer closest to $\sqrt{n}$. Determine

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{1980}}.$$

5. A square is divided into n^2 equal squares and the diagonals of each little square are drawn. Determine, in terms of n, the total number of isosceles right-angled triangles of all sizes.

1981 CELEBRATION OF CHINESE NEW YEAR CONTEST
January 26, 1981

1. What is the coefficient of x^2 when

$$(\cdots(((x-2)^2-2)^2-2)^2-\cdots-2)^2$$

is expanded and like terms are combined?

2. Prove that $1980^{1981^{1982}}+1982^{1981^{1980}}$ is divisible by 1981^{1981}.

3. Let $f(x)=x^{99}+x^{98}+x^{97}+\cdots+x^2+x+1$. Determine the remainder when $f(x^{100})$ is divided by $f(x)$.

4. The base of a tetrahedron is a triangle with side lengths 8, 5 and 5. The dihedral angle between each lateral face and the base is 45°. Determine the volume of the tetrahedron.

5. ABC is a triangle of area 1. D, E and F are the midpoints of BC, CA and AB, respectively. K, L and M are points on AE, CD and BF, respectively. Prove that the area of the intersection of triangles DEF and KLM is at least 1/8.

*   *   *   *   *

The first solutions we print this month were in response to the challenge for the 1984 unsolved problems from *Crux*.

3. [1984: 74] *West Point proposals.*

Determine the maximum area of the convex hull of four circles C_i, $i=1,2,3,4$, each of unit radius, which are placed so that C_i is tangent to C_{i+1} for $i=1,2,3$.

Solution by R.K. Guy, The University of Calgary.

The convex hull is dissected into two parts by the common tangent at the point of contact of C_2 and C_3. If the area of either part is less than that of the other then move C_1 (or C_4) so as to increase the area of that part. So we may assume the symmetry of either of Figure 1 or Figure 2.

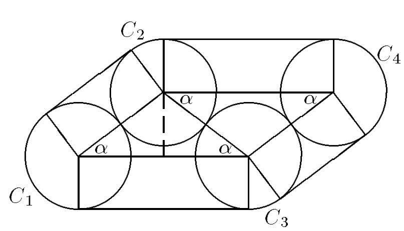

Figure 1

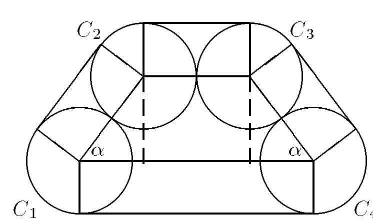

Figure 2

In each figure, the area of the convex hull is made up of sectors of each of the 4 circles (total area π), 4 rectangles and either a parallelogram (Figure 1) or an equilateral trapezium (Figure 2). So the area is either

$$\pi + 2 + 2 + 4\cos\alpha + 4\cos\alpha + (4\cos\alpha)(2\sin\alpha) = \pi + 4 + 8\cos\alpha(1+\sin\alpha)$$

or

$$\pi + 2 + 2 + 2 + (2 + 4\cos\alpha) + \frac{1}{2}(2+2+4\cos\alpha)(2\sin\alpha)$$

$$= \pi + 4 + 4(1+\cos\alpha)(1+\sin\alpha).$$

Clearly the latter yields the larger area (with equality if $\cos\alpha = 1$, i.e., $\alpha = 0$).

Now $(1+\cos\alpha)(1+\sin\alpha)$, where $0 \le \alpha \le \pi/2$, is a maximum **either** when α is at one end of its domain [$\alpha = 0, \alpha = \pi/2$ each yield $\pi + 4 + 4\cdot 2 = \pi + 12$], **or** when $\cos\alpha - \sin\alpha + \cos^2\alpha - \sin^2\alpha = 0$, i.e. $\cos\alpha = \sin\alpha$ or $1 + \cos\alpha + \sin\alpha = 0$. Only the former gives a solution in the domain, namely $\alpha = \pi/4$. This gives

$$\pi + 4 + 4(1 + 1/\sqrt{2})^2 = \pi + 4 + 2(3 + 2\sqrt{2}) = \pi + 10 + 4\sqrt{2} > \pi + 12.$$

So $\alpha = \pi/4$ in Figure 2 yields the maximum area $\pi + 10 + 4\sqrt{2}$.

Similarly, for the minimum we must look at $\pi + 4 + 8\cos\alpha(1+\sin\alpha)$, where $0 \le \alpha \le \pi/3$. Of course, $\alpha = 0$ gives $\pi + 12$. But $\alpha = \pi/3$ gives $\pi + 8 + 2\sqrt{3}$ which is less. After differentiating, $-\sin\alpha - \sin^2\alpha + \cos^2\alpha = 0$, so $2\sin^2\alpha + \sin\alpha = 1$ and $\sin\alpha = 1/2$. For $\alpha = \pi/6$, we have $\pi + 4 + 6\sqrt{3}$ which is a local maximum. Thus the minimum area occurs when $\alpha = \pi/3$ in Figure 1.

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4. [1984: 107] *1983 Chinese Mathematics Olympiad.*

Determine the maximum volume of a tetrahedron whose six edges have lengths $2, 3, 3, 4, 5$, and 5.

Solution by Richard K. Guy, University of Calgary.

A triangle of sides $2, 3, 5$ is degenerate; moreover there is no point at distances $3, 4, 5$ from the vertices of such a triangle. The only possible triangular faces containing edge 2 are thus $233, 234, 245, 255$.

The 2 triangular faces which share edge 2 are	*with opposite edge*	*and the other 2 triangular faces are*
233 & 245	5	345 & 355
233 & 255	4	345 & 345
234 & 255	3	335 & 345

Therefore the only possible tetrahedra are those shown in the following figures.

Figure 1 Figure 2 Figure 3

The volume of the first tetrahedron is smaller than the volume of the second, because in Fig.1 the 4-edge is not perpendicular to the plane of the 233-triangle, whereas it **is** perpendicular in Fig.2.

To see that the volume of the third (Fig.3) is also smaller than that of the second, take the 345-triangle as base. Then, in Fig.3, the angle that the other 3-edge makes with the 345-triangle is less than γ, which is less than β, the angle the 3-edge makes with the 345-triangle in Fig.2, since $\cos\beta = 7/9 < 7/8 = \cos\gamma$.

Thus the maximum volume occurs for the second tetrahedron and is easily calculated to be

$$\frac{1}{3}\cdot\frac{1}{2}(2\sqrt{3^2-1^2})\cdot 4 = \frac{8}{3}\sqrt{2}.$$

5. [1984: 107] *1983 Chinese Mathematics Olympiad.*

Determine

$$\min_{A,B}\max_{0\le x\le 3\pi/2}|\cos^2 x + 2\sin x\cos x - \sin^2 x + Ax + B|.$$

Solution by Richard K. Guy, University of Calgary.

Note that

$$\begin{aligned}\cos^2 x + 2\sin x\cos x - \sin^2 x + Ax + B &= \cos 2x + \sin 2x + Ax + B\\ &= \sqrt{2}\cos(2x-\pi/4) + Ax + B.\end{aligned}$$

The first term, $\sqrt{2}\cos(2x-\pi/4)$, oscillates between $\pm\sqrt{2}$, attaining its maximum and minimum values at $\pi/8, 5\pi/8$, and $9\pi/8$ in the domain $[0, 3\pi/2]$. Any values of A and B other than zero will disturb the symmetry between $\pm\sqrt{2}$ and increase the absolute value to something greater than $\sqrt{2}$. The required minimum is thus $\sqrt{2}$.

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4. [1984: 108] *Austrian-Polish Mathematics Competition 1982.*

N being the set of natural numbers, for every $n \in \mathbf{N}$, let $P(n)$ denote the product of all the digits of n (in base ten). Determine whether or not the sequence $\{x_k\}$, where

$$x_1 \in \mathbf{N}\ ,\ x_{k+1} = x_k + P(x_k),\ k = 1, 2, 3, \dots\ ,$$

can be unbounded (i.e., for every number M, there exists an x_j such that $x_j > M$).

Solution by Richard K. Guy, University of Calgary.

For a d-digit number n, $P(n) \leq 9^d < 10^{d-1}$ for $d \geq 22$. So for $n > 10^{21}$, the left hand digit of x_k increments by at most 1 at each step, so that the leftmost digits will eventually be $10\ldots$ and $P(x_k) = 0$ from then on, and the sequence becomes constant.

It's of interest, perhaps, to ask for the sequence with the largest number of distinct entries. Is it easy to beat 1, 2, 4, 8, 16, 22, 26, 38, 62, 74, 102, ... ?

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5. [1984: 214] *1983 Swedish Mathematical Contest.*

A unit square is to be covered by three congruent disks.

(a) Show that there are disks with radii less than half the diagonal of the square that provide a covering.

(b) Determine the smallest possible radius.

Solution by Richard K. Guy, University of Calgary.

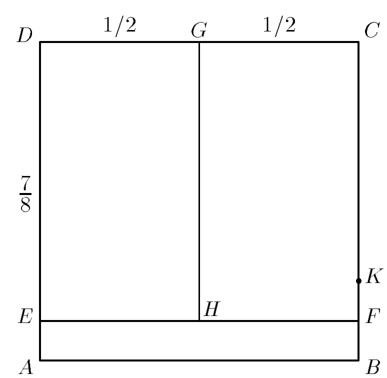
(a) In the covering shown, the radius of the upper disks are

$$\frac{1}{2}\sqrt{(1/2)^2 + (7/8)^2} = \frac{\sqrt{65}}{16}$$

and the radius of the lower disk is

$$\frac{1}{2}\sqrt{1^2 + (1/8)^2} = \frac{\sqrt{65}}{16},$$

and $\sqrt{65}/16 < \sqrt{2}/2$.



(b) Suppose there is a covering with three disks, each of diameter less than $\sqrt{65}/8$. Two of the four corners are covered by the same disk, by the pigeon-hole principle. Without loss of generality we can assume that they are A and B. Then the point E is covered by a different disk, since $EB = \sqrt{65}/8$. And the point G is covered by the third disk since $EG = \sqrt{65}/8$ and $GB > \sqrt{65}/8$. This means that the point F is covered by the second disk, since $FA = FG = \sqrt{65}/8$. Then the point K, where $FK = 1/8$, is covered by the third disk, since $KA > KE = \sqrt{65}/8$. This means D is uncovered, since $DB > DF > DK > \sqrt{65}/8$. This contradicts our assumption that the three disks cover the square. So $\sqrt{65}/8$ is the least possible radius.

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2. [1984: 215] *British Mathematical Olympiad.*

For $0 \leq r \leq n$, let a_n be the number of binomial coefficients $\binom{n}{r}$ which leave remainder 1 on division by 3, and let b_n be the number which leave remainder 2. Prove that $a_n > b_n$ for all positive integers n.

Solution by Andy Liu, University of Alberta.

All congruences ($\equiv$) are taken modulo 3, and we say that a polynomial is *satisfactory* if it has more coefficients congruent to 1 than it has congruent to 2. Let

$$n = 3^k n_k + 3^{k-1} n_{k-1} + \cdots + 3n_1 + n_0$$

be the base 3 representation of n, where $n_i = 0, 1$ or 2 for $0 \le i \le k$. Then we have that

$$\begin{aligned}(1+x)^n &= (1+x)^{3^k n_k}(1+x)^{3^{k-1}n_{k-1}} \cdots (1+x)^{3n_1}(1+x)^{n_0} \\ &\equiv (1+x^{3^k})^{n_k}(1+x^{3^{k-1}})^{n_{k-1}} \cdots (1+x^3)^{n_1}(1+x)^{n_0}.\end{aligned}$$

For $0 \le i \le k$, set

$$F_i(x) = (1+x^{3^i})^{n_i}(1+x^{3^{i-1}})^{n_{i-1}} \cdots (1+x^3)^{n_1}(1+x)^{n_0}.$$

We claim that $F_i(x)$ is satisfactory for $0 \le i \le k$. This is certainly true for $i = 0$. Suppose it holds for some $i < k$, and consider $F_{i+1}(x) = (1+x^{3^{i+1}})^{n_{i+1}} F_i(x)$.

If $n_{i+1} = 0$, we have $F_{i+1}(x) = F_i(x)$, and the result follows from the induction hypothesis.

If $n_{i+1} = 1$, then $F_{i+1}(x) = F_i(x) + x^{3^{i+1}} F_i(x)$. By the induction hypothesis $F_i(x)$ is satisfactory, and so is $x^{3^{i+1}} F_i(x)$. Moreover, since $F_i(x)$ is of degree strictly less than 3^{i+1}, there are no like terms between $F_i(x)$ and $x^{3^{i+1}} F_i(x)$. It follows that $F_{i+1}(x)$ is also satisfactory.

If $n = 2$, then $F_{i+1}(x) = F_i(x) + 2x^{3^{i+1}} F_i(x) + x^{2 \cdot 3^{i+1}} F_i(x)$. Again there are no like terms. Moreover, the numbers of coefficients of $2x^{3^{i+1}} F_i(x)$ congruent to 1 and 2 are respectively equal to the numbers of coefficients of $F_i(x)$ congruent to 2 and 1. Hence $F_{i+1}(x)$ is satisfactory.

It follows that $(1+x)^n \equiv F_k(x)$ is satisfactory, so that $a_n > b_n$ as desired.

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3. [1984: 283] *1984 Annual Greek High School Competition.*

If G is a multiplicative group and a, b, c are elements of G, prove:

(a) If $b^{-1}ab = ac, ac = ca$, and $bc = cb$, then $a^n b = ba^n c^n$ and $(ab)^n = b^n a^n c^{n(n+1)/2}$ for all $n \in \mathbf{N}$.

(b) If $b^{-1}ab = a^k$ where $k \in \mathbf{N}$, then $b^{-1}a^n b^l = a^{nk^l}$ for all $l, n \in \mathbf{N}$.

Solution by Richard K. Guy, University of Calgary.

(a) First $b^{-1}ab = ac$ implies $ab = bac$, which is the required result for $n = 1$. Assume inductively that $a^n b = ba^n c^n$. Then

$$a^{n+1}b = a^n ab = a^n bac = ba^n c^n ac = ba^{n+1}c^{n+1}$$

since a and c commute. (Note that b, c commuting is not needed here.)

Assume inductively that $(ab)^n = b^n a^n c^{n(n+1)/2}$ (which is true for $n = 1$ as noted above). Then, since c commutes with both a and b,

$$\begin{aligned}(ab)^{n+1} &= (ab)^n ab \;=\; b^n a^n c^{n(n+1)/2} ab \\ &= b^n a^n c^{n(n+1)/2} bac \;=\; b^n a^n b c^{n(n+1)/2} ac \\ &= b^n (ba^n c^n) c^{n(n+1)/2} ac \;=\; b^{n+1} a^{n+1} c^{(n+1)(n+2)/2}.\end{aligned}$$

(b) This gave trouble, because it is misprinted (but true for $l = 1$ nevertheless). Note

$$b^{-1}a^nb = (b^{-1}ab)^n = a^{kn}.$$

So $b^{-1}a^nb^l = a^{nk^l}$ for $l = 1$ and all n. It is **not** true for $l > 1$.

However, assume inductively that $b^{-l}a^nb^l = a^{nk^l}$ (true for $l = 1$ and all n). Then

$$b^{-l-1}a^nb^{l+1} = b^{-1}(b^{-l}a^nb^l)b = b^{-1}a^{nk^l}b = a^{nk^{l+1}}.$$

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M856. [1984: 283] *Problems from KVANT.*

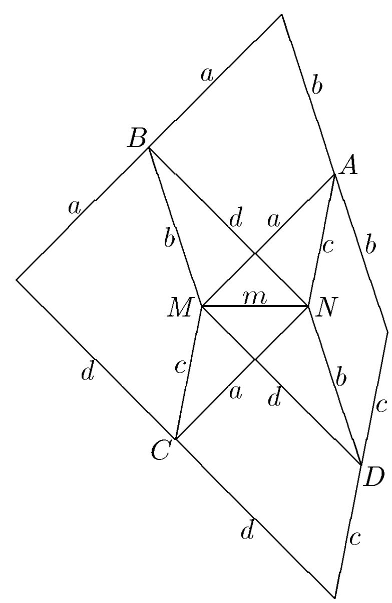
(a) Construct a quadrilateral knowing the lengths of its sides and that of the segment joining the midpoints of the diagonals.

(b) Under what conditions does the problem have a solution?

Solution by Richard K. Guy, University of Calgary.

Suppose first that we are given the lengths of the sides $2a, 2b, 2c, 2d$, *in that cyclic order*, and that m is the length of the join of the midpoints of the diagonals. (Note that $m = 0$ implies that the quadrilateral is a parallelogram and $a = c, b = d$ are necessary; this will be a special case of the general condition.)

(a) The construction is easily discovered from the theorem that the midpoints of the sides of any quadrilateral form a parallelogram. Draw a segment MN of length m. Construct two triangles with sides a, c, m forming a parallelogram $MANC$, say. Similarly use b, d, m to form parallelogram $MBND$. From A draw segments b in either direction, parallel to the sides b of the parallelogram $MBND$. From B draw segments a in either direction parallel to the sides a of the parallelogram $MANC$. From C draw segments d in either direction parallel to the sides d of the parallelogram $MBND$. From D draw segments c in either direction parallel to the sides c of the parallelogram $MANC$. It is easy to prove that the 8 ends of these segments coincide in pairs at the four corners of the required quadrilateral.



(b) The construction will succeed if the parallelogram sides a, b, c, d at M (or at N) are in the required cyclic order. It may not always be possible to achieve this. The necessary and sufficient condition for success is that it is possible to arrange the sides $2a, 2b, 2c, 2d$ in some order so that a, c, m and b, d, m form triangles (possibly degenerate). Readers are encouraged to carry out the construction with sides $10, 10, 12, 12$ and $m = 0, 1, 2, 4, 9, 10, 11$ and see how many different quadrilaterals (which may be nonconvex, or even crossed) can be produced in each case.

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This finishes the "archive" material we have for 1984. We now turn to problems posed in the March 1989 number of the Corner.

3. [1989: 65] *1987 Annual Greek High School Competition.*

Let A be an $n \times n$ matrix such that $A^2 - 3A + 2I = 0$, where I is the identity matrix and 0 the zero matrix. Prove that $A^{2k} - (2^k + 1)A^k + 2^k I = 0$ for every natural number $k \geq 1$.

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

Since $A^{m+n} = A^m A^n$ for all positive inters m, n, we see that if $f(x) = p(x)q(x)$ where $p(x), q(x)$ are polynomials in x, then $f(A) = p(A)q(A)$.

Let $f(x) = x^{2k} - (2^k + 1)x^k + 2^k$. Note that $f(1) = f(2) = 0$. It follows that $f(x) = (x^2 - 3x + 2)q(x)$ for some polynomial $q(x)$. Thus

$$f(A) = (A^2 - 3A + 2I)q(A) = 0q(A) = 0.$$

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3. [1989: 66] *24th Spanish Olympiad-First Round.*

Let C be the set of natural numbers

$$C = \{1, 5, 9, 13, 17, 21, \ldots\}.$$

Say that a number is *prime for C* if it cannot be written as a product of smaller numbers from C.

(a) Show that 4389 is a member of C which can be represented in at least two distinct ways as a product of two numbers prime for C.

(b) Find another member of C with the same property.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Note first that C is closed under multiplication since it consists of all positive numbers of the form $4a + 1$.

(a) Since $4389 = 3 \times 7 \times 11 \times 19$ is a product of four distinct primes of the form $4b + 3$, the product of any two of these primes will have the form $4c + 1$ which clearly cannot be written as a product of smaller numbers from C. Hence 4389 can be written in at least three different ways as a product of two members of C prime for C:

$$4389 = 21 \times 209 = 33 \times 133 = 57 \times 77.$$

In fact, since it is obvious that the product of any three numbers from $3, 7, 11, 19$ is prime for C, there are four other ways of expressing 4389 in the described manner:

$$4389 = 3 \times 1463 = 7 \times 627 = 11 \times 399 = 19 \times 231.$$

(b) By the argument above, to obtain another member of C with the same property it suffices to replace 19 by 23, the next true prime of the form $4k + 3$ and obtain $3 \times 7 \times 11 \times 23 = 5313$.

5. [1989: 67] *24th Spanish Olympiad-First Round.*

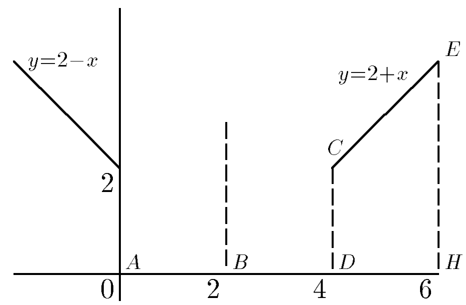
Given the function f defined by $f(x) = \sqrt{4 + \sqrt{16x^2 - 8x^3 + x^4}}$.

(a) Draw the graph of the curve $y = f(x)$.

(b) Find, without the use of integral calculus, the area of the region bounded by the straight lines $x = 0, x = 6, y = 0$ and by the curve $y = f(x)$. Note: all the square roots are non-negative.

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

(a) Note that $f(x) = \sqrt{4 + |x^2 - 4x|}$. If $x > 4$ or $x < 0$ then $f(x) = |x - 2|$. If $0 < x < 4$ then $y = f(x)$ can be written as $(x-2)^2 + y^2 = (2\sqrt{2})^2$, a circle. Thus the graph is as shown.



(b) The area of $\triangle AOB$ and of $\triangle BCD$ is 2, the area of sector ABC is $\frac{1}{4}\pi(2\sqrt{2})^2 = 2\pi$, and the area of trapezoid $CDHE$ is $\frac{1}{2}(2+4) \times 2 = 6$. Thus the area of the region is $8 + 2\pi$.

7. [1989: 67] *24th Spanish Olympiad-First Round.*

Let $I_n = (n\pi - \pi/2, n\pi + \pi/2)$ and let f be the function defined by $f(x) = \tan x - x$.

(a) Show that the equation $f(x) = 0$ has only one root in each interval I_n, $n = 1, 2, 3, \ldots$.

(b) If c_n is the root of $f(x) = 0$ in I_n, find $\lim_{n\to\infty}(c_n - n\pi)$.

Solution by the editors.

(a) This is obvious since $(\tan x - x)' = \sec^2 x - 1 \geq 0$ on I_n.

(b) As n goes to infinity the point of intersection of the line $y = x$ and the graph of $y = \tan x$ in the interval I_n goes off to infinity in each coordinate. Thus the difference $c_n - n\pi$ must go to $\pi/2$.

[*Editor's note.* This solution was adapted from the rather more detailed one submitted by Seung-Jin Bang, Seoul, Republic of Korea.]

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1. [1989: 67] *24th Spanish Olympiad.*

Fifteen problems, numbered 1 through 15, are posed on a certain examination. No student answers two consecutive problems correctly. If 1600 candidates sit the test, must at least two of them answer each question in the same way?

Solutions by John Morvay, Springfield, Missouri, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The answer is yes, assuming each question has only two possible answers, right and wrong, and assuming no student leaves any question unanswered. First note that the answer pattern of each student corresponds to a sequence of length 15, consisting of the two symbols R and W in which there are no two consecutive R's. Clearly the maximum number of R's is 8. For each $0 \leq k \leq 8$, the number of those sequences with exactly k R's is easily seen to be $\binom{15-k+1}{k} = \binom{16-k}{k}$ since each such sequence is equivalent to a way of inserting k bookmarks in the $16-k$ slots (including the two "ends") between $15-k$ books. Therefore the total number of sequences with the described property is:

$$\sum_{k=0}^{8}\binom{16-k}{k} = 1+15+91+286+495+462+210+36+1 = 1597 < 1600.$$

This shows that at least two students must have identical answer patterns. Indeed if there are n questions the maximum possible number of students, no two with the same answer pattern, is

$$\sum_{k=0}^{\left[\frac{n+1}{2}\right]}\binom{n-k+1}{k}.$$

2. [1989: 68] *24th Spanish Olympiad.*

Let f be a continuous function on $\mathbf{R}$ such that
(i) $f(n)=0$ for every integer n, and
(ii) if $f(a)=0$ and $f(b)=0$ then $f(\frac{a+b}{2})=0$, with $a \neq b$.
Show that $f(x)=0$ for all real x.

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

If n is an integer then $f(n/2)=0$. By induction on m, we have $f(n/2^m)=0$. Let s and ϵ be arbitrary real numbers. Since f is continuous at s, there is $\delta > 0$ such that $|f(s)-f(x)| < \epsilon$ whenever $|s-x| < \delta$. Since s has a 2-adic expansion, there is $n/2^m$ such that $|s-n/2^m| < \delta$. We now have $|f(s)-f(n/2^m)| = |f(s)| < \epsilon$. Hence $f(s)=0$. (This is, of course, just the standard argument.)

7. [1989: 68] *24th Spanish Olympiad.*

Solve the following system of equations in the set of complex numbers:

$$|z_1|=|z_2|=|z_3|=1,$$
$$z_1+z_2+z_3=1,$$
$$z_1z_2z_3=1.$$

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

Let $\overline{z}$ denote the complex conjugate of z. We have $\overline{z}_i = 1/z_i$ for $i=1,2,3$. It follows that

$$z_1z_2+z_2z_3+z_3z_1 = \frac{1}{z_3}+\frac{1}{z_1}+\frac{1}{z_2} = \overline{z}_3+\overline{z}_1+\overline{z}_2 = \overline{z_1+z_2+z_3} = 1.$$

Consider the cubic polynomial

$$(x-z_1)(x-z_2)(x-z_3) = x^3-x^2+x-1 = (x-1)(x^2+1).$$

Since $1, \pm i$ are the roots we have that z_1, z_2, z_3 are equal to $1, i, -i$ in some order.

* * * * *

This completes the solutions submitted for problems from the March 1989 number and this is all the space we have this month. Send me your contests!

MINI-REVIEWS

by ANDY LIU

BOOKS FROM DOVER PUBLICATIONS, INC.

The majority of Dover's publications are reprints of excellent (otherwise, why do it?) books that are no longer available in other formats. The new editions are usually paperbound and inexpensive (averaging about $5.00 U.S. each). Often, errors in the original versions are corrected, and new material appended. While Dover has a large selection of titles in main-stream mathematics (as well as in many other areas, academic or otherwise), we will focus on the best of its line on popular mathematics. All are paperbacks.

Challenging Mathematical Problems with Elementary Solutions I, by A.M. Yaglom and I.M. Yaglom, 1987. (231 pp.)

This is one of the finest collections of problems in elementary mathematics. The 100 problems in combinatorial analysis and probability theory are all easy to understand, but some are not easy to solve, even though no advanced mathematics is required.

Challenging Mathematical Problems with Elementary Solutions II, by A.M. Yaglom and I.M. Yaglom, 1987. (214 pp.)

In this second volume, 74 problems are selected from various branches of mathematics, in particular, number theory and combinatorial geometry. This book and the earlier volume is a must for every school library.

Mathematical Bafflers, edited by A.F. Dunn, 1980. (337 pp.)

The bafflers in this book originally appeared as a most successful weekly corporate advertisement in technical publications. They are contributed by the readers, with a consequent diversity in their levels of sophistication. Some require almost no mathematics while others are quite demanding. However, there is a beautiful idea behind each baffler, which is compactly stated and accompanied by a cartoon.

Second Book of Mathematical Bafflers, edited by A.F. Dunn, 1983. (186 pp.)

This second collection of bafflers, like the earlier volume, is organized by chapters, each dealing with one area of mathematics. These include algebra, geometry, Diophantine problems and other number theory problems, logic, probability and "insight".

Ingenious Mathematical Problems and Methods, by L.A. Graham, 1959. (237 pp.)

The 100 problems in this book originally appeared in the "Graham Dial", a publication circulated among engineers and production executives. They are selected from areas not commonly included in school curricula, and have new and unusual twists that call for ingenious solutions.

The Surprise Attack in Mathematical Problems, by L.A. Graham, 1968. (125 pp.)

These 52 problems are selected from the "Graham Dial" on the criterion that the best solutions are not the ones the original contributors had in mind. The reader will enjoy the elegance of the unexpected approach. Like the earlier volume, the book includes a number of illustrated Mathematical Nursery Rhymes.

One Hundred Problems in Elementary Mathematics, by H. Steinhaus, 1979. (174 pp.)

The one hundred problems cover the more traditional areas of number theory, algebra, plane and solid geometry, as well as a host of practical and non-practical problems. There are also thirteen problems without solutions, some but not all of these actually having known solutions. The unsolved problems are not identified in the hope that the reader will not be discouraged from attempting them.

Fifty Challenging Problems in Probability with Solutions, by F. Mosteller, 1987. (88 pp.)

This book actually contains fifty-six problems, each with an interesting story-line. There are the familiar "gambler's ruin" and "birthday surprises" scenarios, but with new twists. Others are unconventional, including one which turns out to be a restatement of Fermat's Last Theorem.

Mathematical Quickies, by C. Trigg, 1985. (210 pp.)

This book contains two hundred and seventy problems. Each is chosen because there is an elegant solution. Classification by subject is deliberately avoided, nor are the problems graduated in increasing level of difficulty. This encourages the reader to explore each problem with no preconceived idea of how it should be approached.

Entertaining Mathematical Puzzles, by Martin Gardner, 1986. (112 pp.)

The master entertains with thirty-nine problems and twenty-eight quickies, covering arithmetic, geometry, topology, probability and mathematical games. There is a brief introduction to the basic ideas and techniques in each section.

Mathematical Puzzles of Sam Loyd I, edited by Martin Gardner, 1959. (167 pp.)

Sam Loyd is generally considered the greatest American puzzlist. He had a knack of posing problems in a way which attracts the public's eye. Many of the one hundred and seventeen problems in this book had been used as novelty advertising give-aways.

Mathematical Puzzles of Sam Loyd II, edited by Martin Gardner, 1960. (177 pp.)

This book contains one hundred and sixty-six problems, most of which are accompanied by Loyd's own illustrations, as is the case with the earlier volume. The two books represent the majority of the mathematical problems in the mammoth "Cyclopedia" by Sam Loyd, published after his death.

Amusements in Mathematics, by H.E. Dudeney, 1970. (258 pp.)

Henry Ernest Dudeney, a contemporary of Sam Loyd, is generally considered the greatest English puzzlist, and a better mathematician than Loyd. This book contains four hundred and thirty problems, representing only part of Dudeney's output. There are plenty of illustrations in the book.

Mathematical Puzzles for Beginners and Enthusiasts, by G. Mott-Smith, 1954. (248 pp.)

This book contains one hundred and eighty-nine problems in arithmetic, logic, algebra, geometry, combinatorics, probability and mathematical games. They are both instructive and entertaining.

Mathematical Recreations and Essays, by W.R. Ball and H.S.M. Coxeter, 1988. (418 pp.)

This is the foremost single-volume classic of popular mathematics. Written by two distinguished mathematicians, it covers a variety of topics in great detail. After arithmetical and geometrical recreations, it moves on to polyhedra, chessboard recreations, magic squares, map-colouring problems, unicursal problems, Kirkman's schoolgirls problem, the three classical geometric construction problems, calculating prodigies, cryptography and cryptanalysis.

Mathematical Recreations, by M. Kraitchik, 1953. (330 pp.)

This is a revision of the author's original work in French. It covers more or less the same topics as "Mathematical Recreations and Essays". There is a chapter on ancient and curious problems from various sources.

The Master Book of Mathematical Recreation, by F. Schuh, 1968. (430 pp.)

This is a translation of the author's original work in German. Four of the fifteen chapters are devoted to the analysis of mathematical games. The remaining ones deal with puzzles of various kinds. General hints for solving puzzles are given in the introductory chapter. The last chapter is on puzzles in mechanics.

Puzzles and Paradoxes, by T.H. O'Beirne, 1984. (238 pp.)

Like Martin Gardner's series, this book is an anthology of the author's column in *New Scientist*. It consists of twelve largely independent articles.

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **August 1, 1991,** *although solutions received after that date will also be considered until the time when a solution is published.*

1601. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a right-angled triangle with the right angle at A. Let D be the foot of the perpendicular from A to BC, and let E and F be the intersections of the bisector of $\angle B$ with AD and AC respectively. Prove that $\overline{DC} > 2\overline{EF}$.

1602. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Suppose $x_1, x_2, \ldots, x_n \in [0,1]$ and $\sum_{i=1}^{n} x_i = m + r$ where m is an integer and $r \in [0,1)$. Prove that

$$\sum_{i=1}^{n} x_i^2 \leq m + r^2.$$

1603. *Proposed by Clifford Gardner, Austin, Texas, and Jack Garfunkel, Flushing, N.Y.*

Given is a sequence $,_1, ,_2, \ldots$ of concentric circles of increasing and unbounded radii and a triangle $A_1B_1C_1$ inscribed in $,_1$. Rays $\overrightarrow{A_1B_1}, \overrightarrow{B_1C_1}, \overrightarrow{C_1A_1}$ are extended to intersect $,_2$ at B_2, C_2, A_2, respectively. Similarly, $\Delta A_3B_3C_3$ is formed in $,_3$ from $\Delta A_2B_2C_2$, and so on. Prove that $\Delta A_nB_nC_n$ tends to the equilateral as $n \to \infty$, in the sense that the angles of $\Delta A_nB_nC_n$ all tend to 60°.

1604. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Ever active Pythagoras recently took a stroll along a street where only Pythagoreans lived. He was happy to notice that the houses on the left side were numbered by squares of consecutive natural numbers while the houses on the right were numbered by fourth powers of consecutive natural numbers, both starting from 1. Each side had the same (reasonably large) number of houses. At some point he noticed a visitor. "It is awesome!" said the visitor on encountering Pythagoras. "Never did I see houses numbered this way." In a short discussion that followed, the visitor heard strange things about numbers. And when it was time to part, Pythagoras asked "How many houses did you see on each side of the street?" and soon realized that counting was an art that the visitor had never mastered. "Giving answers to my questions is not my habit", smilingly Pythagoras continued. "Go to a *Crux* problem solver, give the clue that the sum of the house numbers on one side is a square multiple of the corresponding sum on the other side and seek help."

1605. *Proposed by M.S. Klamkin and Andy Liu, University of Alberta.*

ADB and AEC are isosceles right triangles, right-angled at D and E respectively, described outside ΔABC. F is the midpoint of BC. Prove that DFE is an isosceles right-angled triangle.

1606*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For integers $n \geq k \geq 1$ and real x, $0 \leq x \leq 1$, prove or disprove that

$$\left(1 - \frac{x}{k}\right)^n \geq \sum_{j=0}^{k-1} \left(1 - \frac{j}{k}\right) \binom{n}{j} x^j (1-x)^{n-j}.$$

1607. *Proposed by Peter Hurthig, Columbia College, Burnaby, B.C.*

Find a triangle such that the length of one of its internal angle bisectors (measured from the vertex to the opposite side) equals the length of the external bisector of one of the other angles.

1608. *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Suppose n and r are nonnegative integers such that no number of the form $n^2+r-k(k+1)$, $k=1,2,\ldots$, equals -1 or a positive composite number. Show that $4n^2+4r+1$ is 1, 9, or prime.

1609. *Proposed by John G. Heuver, Grande Prairie Composite H.S., Grande Prairie, Alberta.*

P is a point in the interior of a tetrahedron $ABCD$ of volume V, and F_a, F_b, F_c, F_d are the areas of the faces opposite vertices A, B, C, D, respectively. Prove that

$$PA \cdot F_a + PB \cdot F_b + PC \cdot F_c + PD \cdot F_d \geq 9V.$$

1610. *Proposed by P. Penning, Delft, The Netherlands.*

Consider the multiplication $d \times dd \times ddd$, where $d < b-1$ is a nonzero digit in base b, and the product (base b) has six digits, all less than $b-1$ as well. Suppose that, when d and the digits of the product are all increased by 1, the multiplication is still true. Find the lowest base b in which this can happen.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

689. [1981: 276; 1982: 307; 1983: 144] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Let m_a, m_b, m_c denote the lengths of the medians to sides a, b, c, respectively, of triangle ABC, and let M_a, M_b, M_c denote the lengths of these medians extended to the circumcircle of the triangle. Prove that

$$\frac{M_a}{m_a} + \frac{M_b}{m_b} + \frac{M_c}{m_c} \geq 4.$$

IV. *Generalization by Dragoljub M. Milošević, Pranjani, Yugoslavia.*

In his solution [1982: 308–309], M.S. Klamkin showed that the problem is equivalent to

$$\sum \frac{a^2}{2(b^2+c^2)-a^2} \geq 1$$

where the sum is cyclic over a, b, c. He also proved the related result

$$\sum \frac{a}{k(b+c)-a} \geq \frac{3}{2k-1}$$

where $k \geq 1$, and suggested the more general problem of finding all $k \geq 2$ such that

$$\sum \frac{a^2}{k(b^2+c^2)-a^2} \geq \frac{3}{2k-1},$$

or even all k and n such that

$$\sum \frac{a^n}{k(b^n+c^n)-a^n} \geq \frac{3}{2k-1}.$$

Here we prove the inequality

$$\sum \frac{a^{\lambda n}}{k(b^n+c^n)-a^n} \geq \frac{3^{2-\lambda}(a^n+b^n+c^n)^{\lambda-1}}{2k-1} \tag{1}$$

for $\lambda = 1$ or $\lambda \geq 2$, and for $k \geq 2^{n-1}$ where $n \geq 1$, generalizing the above inequalities.

Start with the function

$$f(x) = \frac{x^\lambda}{p-qx} \quad , \quad 0 < x < p/q,$$

where $p > 0, q > 0, \lambda \in \{1\} \cup [2, \infty)$. Since

$$f''(x) = \frac{x^{\lambda-2}}{(p-qx)^3}[(\lambda-1)(\lambda-2)q^2x^2 - 2pq\lambda(\lambda-2)x + p^2\lambda(\lambda-1)] > 0$$

for $\lambda \geq 2$ and also for $\lambda = 1$, function f is convex, so for $0 < x_i < p/q$

$$\sum_{i=1}^{3} \frac{x_i^\lambda}{p-qx_i} = \sum_{i=1}^{3} f(x_i) \geq 3f\left(\frac{x_1+x_2+x_3}{3}\right) = \frac{3^{2-\lambda}(x_1+x_2+x_3)^\lambda}{3p-q(x_1+x_2+x_3)}. \tag{2}$$

Putting in (2)

$$p = (a^n+b^n+c^n)k \quad , \quad q = k+1 \quad , \quad x_1 = a^n,\ x_2 = b^n,\ x_3 = c^n,$$

we obtain the desired inequality (1). Note that, for example, $x_1 < p/q$ is equivalent to

$$a^n < k(b^n+c^n),$$

so since $k \geq 2^{n-1}$ it is enough to prove that

$$(b+c)^n < 2^{n-1}(b^n+c^n),$$

which is true by the convexity of x^n for $n \geq 1$.

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1411. [1989: 47; 1990: 92] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

ΔABC is acute angled with sides a, b, c and has circumcircle , with centre O. The inner bisector of $\angle A$ intersects , for the second time in A_1. D is the projection on AB of A_1. L and M are the midpoints of CA and AB respectively. Show that

(i) $AD = \frac{1}{2}(b + c)$;

(ii) $A_1D = OM + OL$.

II. *Comment by Toshio Seimiya, Kawasaki, Japan.*

[This is in response to a question of the editor [1990:93].]

Part (i) remains true for nonacute triangles. Here is a proof for all triangles. Letting E be the foot of the perpendicular from A_1 to AC, we get $AD = AE$ and $A_1D = A_1E$. Because $BA_1 = CA_1$, we then have $\Delta A_1DB \equiv \Delta A_1EC$, and therefore $BD = CE$. Thus

$$b + c = AB + AC = AD + AE = 2AD,$$

and (i) follows.

Part (ii) does not hold for all triangles. If $\angle B > 90^\circ$ we get $A_1D = OM - OL$, and if $\angle C > 90^\circ$ we get $A_1D = OL - OM$.

By using the relation (ii) (for acute triangles) we have an alternate proof of the well known theorem: *if ΔABC is an acute triangle with circumcentre O, circumradius R, and inradius r, and L, M, N are the midpoints of the sides, then*

$$OL + OM + ON = R + r.$$

(N.A. Court, *College Geometry*, p. 73, Thm. 114). [*Editor's note*: this theorem was also used by Seimiya in his proof of *Crux* 1488, this issue.]

Let I be the incenter of ΔABC, and let T, S be the feet of the perpendiculars from I to AD, A_1D respectively. Then we get $SD = IT = r$. It is well known that $A_1I = A_1C$, and because

$$\angle A_1IS = \angle A_1AB = \angle A_1CB$$

and

$$\angle ISA_1 = \angle CNA_1 = 90^\circ,$$

we get $\Delta A_1IS \equiv \Delta A_1CN$, from which we have $A_1S = A_1N$. Using (ii) we have

$$\begin{aligned} OL + OM + ON &= A_1D + ON = A_1S + SD + ON \\ &= A_1N + r + ON = R + r. \end{aligned}$$

Another solution was received from K.R.S. SASTRY, Addis Ababa, Ethiopia, in which he also shows that converses of (i) *and* (ii) *need not hold, and finds similar formulae in the case that AA_1 is the external bisector of $\angle A$.*

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1432. [1989: 110; 1990: 180] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

If the Nagel point of a triangle lies on the incircle, prove that the sum of two of the sides of the triangle equals three times the third side.

IV. *Comment and solution by Dan Sokolowsky, Williamsburg, Virginia.*

This is in response to L.J. Hut's claim [1990: 182] that, if D is the point at which the incircle of ΔABC touches AB, then the incircles of triangles ABC and $A'B'C'$ touch at D. The claim is true provided that some additional assumption (such as (2) below) is made.

First, it is easy to see (referring to the figure on [1990: 181]) that this claim is equivalent to

$$NI \perp AB \text{ at } D. \tag{1}$$

We show, assuming that the Nagel point of ΔABC lies on its incircle, that (1) is equivalent to: *AB is the shortest side of ΔABC*, that is, to

$$c \leq a, b \tag{2}$$

(which, incidentally, would justify Hut's selection of AB over the other edges of ΔABC).

In the figure, let w denote the incircle of ΔABC, touching AB at D, CA at Z, CB at Y. Let P, Q be points on CA, CB respectively such that $PQ \parallel AB$ and PQ touches w, say at F. Obviously $FI \perp PQ$, hence $FI \perp AB$. Thus to show (2) implies (1) it will suffice to show that (2) implies F is the Nagel point N of ΔABC.

Let CF meet w again at F' and AB at V. Note that w is the excircle of ΔCPQ on side PQ, hence the Nagel point of ΔCPQ lies on CF. Then, since $\Delta ABC \sim \Delta CPQ$, the Nagel point N of ΔABC lies on CV. By hypothesis, it also lies on w, hence it is either F or F'. We can assume that F' lies on the arc DZ. Then BF' meets AZ at a point X. If F' were the Nagel point of ΔABC we would then have

$$s - c = AX < AZ = s - a$$

(s the semiperimeter), which implies $a < c$, contradicting (2). It follows that F' cannot be the Nagel point of ΔABC, which must then be F, so (1) follows. Conversely, if (1) holds, then by the preceding argument NI is perpendicular to the shortest side of ΔABC, which must therefore be AB, so (2) holds.

A simple proof of the problem could now go as follows. Parts marked in the adjoining figure have the same meaning as before, but we assume (2), so that by the above the point marked N is the Nagel point of ΔABC. Let AN meet BC at T, and let s' be the semiperimeter of ΔCPQ. We then have

$$\begin{aligned} 2s' &= (CP + PN) + (NQ + QC) \\ &= CZ + CY \;=\; 2(s - c). \end{aligned}$$

Since $\Delta CPQ \sim \Delta CAB$,

$$\frac{CN}{CV} = \frac{s'}{s} = \frac{s-c}{s}$$

and hence

$$\frac{CN}{NV} = \frac{s-c}{c}.$$

Since N is the Nagel point of ΔABC,

$$AV = s - b\ ,\ BT = s - c\ ,\ TC = s - b.$$

By Menelaus applied to ΔCVB,

$$1 = \frac{CN}{NV} \cdot \frac{AV}{AB} \cdot \frac{BT}{TC} = \frac{s-c}{c} \cdot \frac{s-b}{c} \cdot \frac{s-c}{s-b} = \frac{(s-c)^2}{c^2}.$$

Thus $s - c = c$, which implies $a + b = 3c$.

Sokolowsky also pointed out a typo on [1990: 182], *line* 2: $DG = 2GD'$ *should read* $2DG = GD'$.

L.J. Hut submitted a further clarification of his solution, showing not only that $IN \perp AB$ *implies* $a + b = 3c$, *but also that* $IN \parallel AB$ *implies* $a + b = 2c$. *Interesting! Any comments?*

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1481. *Proposed by J.T. Groenman, Arnhem, The Netherlands, and D.J. Smeenk, Zaltbommel, The Netherlands.*

Let A, B, C be points on a fixed circle with B, C fixed and A variable. Points D and E are on segments BA and CA, respectively, so that $\overline{BD} = m$ and $\overline{CE} = n$ where m and n are constants. Points P and Q are on BC and DE, respectively, so that

$$\overline{BP} : \overline{PC} = \overline{DQ} : \overline{QE} = k,$$

also a constant. Prove that the length of PQ is a constant. (This is not a new problem. A reference will be given when the solution is published.)

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

Draw the lines XB and YP parallel to CE and of the same length, so that X, Y, E are collinear and $XE \| BC$. Also find Q on DE so that $YQ \| XD$. Then

$$\frac{DQ}{QE} = \frac{XY}{YE} = \frac{BP}{PC} = k, \qquad (1)$$

so Q is as defined in the problem. We get the *fixed* triangle BDX with sides $BD = m$ and $BX = CE = n$ and included angle α (since A lies on a fixed circle). Thus for every position of A the side XD is fixed. From (1) the side QY is fixed. Since also $PY = CE = n$ and $\angle PYQ = \angle BXD$ is fixed, side PQ is constant.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposers.

Although nobody mentioned it, it appears that point A must be restricted to one of the arcs BC of the circle.

The problem was found by the proposers in Journal de Mathématiques Élémentaires (1912).

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1482. *Proposed by M.S. Klamkin, University of Alberta.*

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are vectors such that

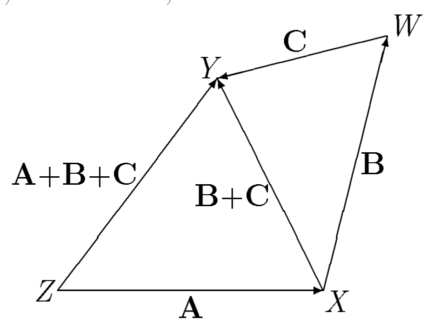
$$|\mathbf{A}| = |\mathbf{B}| = |\mathbf{C}| = |\mathbf{A} + \mathbf{B} + \mathbf{C}|,$$

prove that

$$|\mathbf{B} \times \mathbf{C}| = |\mathbf{A} \times (\mathbf{B} + \mathbf{C})|.$$

I. *Solution by Hans Lausch, Monash University, Melbourne, Australia.*

Let W, X, Y, Z be points in $\mathbf{R}^3$ such that $\overrightarrow{ZX} = \mathbf{A}$, $\overrightarrow{XW} = \mathbf{B}$, and $\overrightarrow{WY} = \mathbf{C}$. Then



$$ZX = ZY = WX = WY,$$

so

$$\Delta XYZ \cong \Delta XYW.$$

Therefore

$$|\mathbf{B} \times \mathbf{C}| = 2 \cdot \text{area}(XYW) = 2 \cdot \text{area}(XYZ) = |\mathbf{A} \times (\mathbf{B} + \mathbf{C})|.$$

II. *Solution by Seung-Jin Bang, Seoul, Republic of Korea.*

It suffices to prove the result if

$$|\mathbf{A}| = |\mathbf{B}| = |\mathbf{C}| = |\mathbf{A} + \mathbf{B} + \mathbf{C}| = 1.$$

Then

$$\begin{aligned} 1 = |\mathbf{A} + \mathbf{B} + \mathbf{C}|^2 &= \mathbf{A}^2 + 2\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) + \mathbf{B}^2 + 2\mathbf{B} \cdot \mathbf{C} + \mathbf{C}^2 \\ &= 3 + 2\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) + 2\mathbf{B} \cdot \mathbf{C}, \end{aligned}$$

so

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = -(\mathbf{B} \cdot \mathbf{C} + 1).$$

It follows from

$$(\mathbf{u} \times \mathbf{v})^2 = \mathbf{u}^2\mathbf{v}^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

(twice) that

$$\begin{aligned} |\mathbf{A} \times (\mathbf{B} + \mathbf{C})|^2 &= \mathbf{A}^2(\mathbf{B} + \mathbf{C})^2 - (\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}))^2 \\ &= (\mathbf{B} + \mathbf{C})^2 - (\mathbf{B} \cdot \mathbf{C} + 1)^2 \\ &= 1 - (\mathbf{B} \cdot \mathbf{C})^2 = |\mathbf{B} \times \mathbf{C}|^2, \end{aligned}$$

which completes the proof.

Also solved by DUANE M. BROLINE, Eastern Illinois University, Charleston; JORDI DOU, Barcelona, Spain; G.P. HENDERSON, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; CHRIS WILDHAGEN, Breda, The Netherlands; and the proposer.

The solutions of Dou, Henderson, and McCallum were similar to solution I.

The proposer gave the following geometric interpretation. Consider the transformation

$$\mathbf{X} = \frac{\mathbf{B} \times \mathbf{C}}{[\mathbf{ABC}]}, \quad \mathbf{Y} = \frac{\mathbf{C} \times \mathbf{A}}{[\mathbf{ABC}]}, \quad \mathbf{Z} = \frac{\mathbf{A} \times \mathbf{B}}{[\mathbf{ABC}]},$$

where $[\mathbf{ABC}] = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. *Then it is known (e.g., Spiegel,* Vector Analysis, *Schaum,* 1959, *Chapter* 2, *exercises* 53(c) *and* 103*) that, reciprocally,*

$$\mathbf{A} = \frac{\mathbf{Y} \times \mathbf{Z}}{[\mathbf{XYZ}]} \quad , \quad \mathbf{B} = \frac{\mathbf{Z} \times \mathbf{X}}{[\mathbf{XYZ}]} \quad , \quad \mathbf{C} = \frac{\mathbf{X} \times \mathbf{Y}}{[\mathbf{XYZ}]} \quad ,$$

and

$$[\mathbf{XYZ}] \cdot [\mathbf{ABC}] = 1.$$

Substituting for $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *in the original problem, one gets the dual problem: if* $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ *are vectors such that*

$$|\mathbf{Y} \times \mathbf{Z}| = |\mathbf{Z} \times \mathbf{X}| = |\mathbf{X} \times \mathbf{Y}| = |(\mathbf{Y} \times \mathbf{Z}) + (\mathbf{Z} \times \mathbf{X}) + (\mathbf{X} \times \mathbf{Y})|,$$

then (by symmetry)

$$|\mathbf{X}| = |\mathbf{Y} - \mathbf{Z}| \quad , \quad |\mathbf{Y}| = |\mathbf{Z} - \mathbf{X}| \quad , \quad |\mathbf{Z}| = |\mathbf{X} - \mathbf{Y}| \quad .$$

Now consider a tetrahedron $PXYZ$ *where* $\overrightarrow{PX} = \mathbf{X}, \overrightarrow{PY} = \mathbf{Y}, \overrightarrow{PZ} = \mathbf{Z}$. *Then the above shows that if the four faces of a tetrahedron have equal areas, the tetrahedron must be isosceles, i.e., opposite pairs of edges are congruent. For a geometric proof, see N. Altshiller Court,* Modern Pure Solid Geometry, *Macmillan, N.Y.,* 1935, *Corollary* 307.

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1483. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be a triangle inscribed in a triangle ABC, so that $A' \in BC, B' \in CA, C' \in AB$, and so that $A'B'C'$ and ABC are directly similar.

(a) Show that, if the centroids G, G' of the triangles coincide, then either the triangles are equilateral or A', B', C' are the midpoints of the sides of ΔABC.

(b) Show that if either the circumcenters O, O' or the incenters I, I' of the triangles coincide, then the triangles are equilateral.

Solution by Jordi Dou, Barcelona, Spain.

When ABC is equilateral the triangles $A'B'C'$ are concentric with ABC and all of G', O', I' coincide with $O = G = I$. Hereafter we suppose that ABC is not equilateral.

Let T'_0 be the triangle whose vertices are the midpoints A'_0, B'_0, C'_0 of BC, CA, AB, respectively. The perpendicular bisectors of the sides of ABC (through A'_0, B'_0, C'_0) concur at the circumcentre O of ABC. Any triangle $T' = A'B'C'$ as described in the problem can be obtained by applying a rotation of centre O and angle x, followed by a homothety of centre O and ratio $r = 1/\cos x$, to triangle T'_0. [*Editor's note.* Can someone supply a reference?] The vertices A'_x, B'_x, C'_x of the resulting triangle T'_x will be on the sides of ABC. Since T'_0 is similar to ABC, T'_x will also be similar to ABC for every x. Note that O is the orthocentre H'_0 of T'_0 and thus also the orthocentre H'_x of every T'_x. Let G'_x be the centroid of T'_x. From $G'_xO = (1/\cos x)G'_0O$ and $\angle G'_0OG'_x = x$ it follows that $G'_xG'_0 \perp G'_0O$. Thus the locus of the centroids G'_x is the line through G'_0 perpendicular to OG'_0. Analogously the locus of the circumcentres O'_x and of the incentres I'_x of the triangles T'_x are the lines through O'_0 and I'_0 perpendicular to OO'_0 and OI'_0 respectively.

Since $G = G'_0$, $G = G'_x$ only for $x = 0$; this solves (a). Since $O = H'_0 \neq O'_0$ unless T'_0 (i.e., ABC) is equilateral, O cannot coincide with O'_x; this solves the first part of (b). Finally, if $I = I'_x$ then, since $G = G'_0$ is on the line II'_0, $\angle HIG = \angle H'_0 I'_0 G'_0 = 90°$.

Editor's note. Dou claimed not to have finished the proof of the impossibility of $I = I'_x$, but it now follows immediately from the known result that $GH^2 \geq GI^2 + IH^2$, with equality only for the equilateral triangle. See p. 288 of Mitrinović, Pečarić, and Volenec, *Recent Advances in Geometric Inequalities*, or the solution of *Crux* 260 [1978: 58].

Also solved by the proposer. Part (a) only was solved by JILL HOUGHTON, Sydney, Australia.

For part (a), the proposer simply applied his two earlier problems Crux 1464 [1990: 282] *and* Crux 1455 [1990: 249].

* * * * *

1484. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $0 < r, s, t \leq 1$ be fixed. Show that the relation

$$r \cot rA = s \cot sB = t \cot tC$$

holds for exactly one triangle ABC, and that this triangle maximizes the expression

$$\sin rA \sin sB \sin tC$$

over all triangles ABC.

Solution by Murray S. Klamkin, University of Alberta.

At most one of the angles rA, sB, tC can be greater than $\pi/2$ so that, by the equality conditions, none of them are greater than $\pi/2$. Consequently the given cotangents are monotonic in their angle argument. Now assume that A, B, C and A', B', C' are different solutions. Since $A' + B' + C' = A + B + C = \pi$, one pair of angles from A', B', C' must be bigger and smaller than the corresponding pair in A, B, C. This gives a contradiction since the cotangents are monotonic. Consequently ABC is unique.

To maximize $\sin rA \sin sB \sin tC$ we take logs and use Lagrange multipliers with Lagrangian

$$\mathcal{L} = \log \sin rA + \log \sin sB + \log \sin tC - \lambda(A + B + C).$$

Then

$$\frac{\partial \mathcal{L}}{\partial A} = \frac{\partial \mathcal{L}}{\partial B} = \frac{\partial \mathcal{L}}{\partial C} = 0$$

yields the given cotangent relations for the maximum. On the boundary, i.e. for A or B or C equal to 0, we obtain the minimum value 0.

More generally, a similar argument goes through to maximize

$$\sin^u rA \, \sin^v sB \, \sin^w tC$$

with the additional condition $u, v, w > 0$. Here the maximizing equations are

$$ur \cot rA = vs \cot sB = wt \cot tC.$$

Also, a similar argument for maximizing

$$\cos^u rA \, \cos^v sB \, \cos^w tC$$

does not go through the same way. Here the extremal equations are

$$ur \tan rA = vs \tan sB = wt \tan tC.$$

However, we now have to check the boundary. This entails setting one and then two of A, B, C equal to 0. Then we have to decide the absolute minimum and maximum from these seven possibilities. We leave this as an open problem.

Also solved by the proposer, who mentions that the problem contains as special cases the items 2.10–2.13 *of Bottema et al,* Geometric Inequalities*.*

By "exactly one" triangle (the editor's wording) was meant of course "up to similarity"!

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1485. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

From a deck of 52 cards, 13 are chosen. Replace one of them by one of the remaining 39 cards. Continue the process until the initial set of 13 cards reappears. Is it possible that all the $\binom{52}{13}$ combinations appear on the way, each exactly once?

Comment by Stanley Rabinowitz, Westford, Massachusetts.

It is known that the subsets of size k from a set of size n can be arranged in a circular list such that adjacent sets in the list differ by the replacement of one element by another. (The subsets are said to be in *revolving door order.*) A reference is [1] in which an algorithm for forming such a list is given (not just an existence proof).

The idea behind the algorithm is as follows. If $A(m, l)$ denotes a list of all the l-subsets of $\{1, 2, \ldots, m\}$ arranged in revolving door order beginning with $\{1, 2, \ldots, l\}$ and ending with $\{1, 2, \ldots, l-1, m\}$, then it can be shown that

$$A(n, k) = A(n-1, k) \ , \ \overline{A(n-1, k-1)} \times \{n\},$$

where the bar means that the order of the list is reversed and the cross means that the element n is appended to each subset in the list. It is easy to check that if $A(n-1, k)$ and $A(n-1, k-1)$ are in revolving door order, then so is $A(n, k)$. It follows by induction that the list $A(n, k)$ exists for each n and k.

Reference:

[1] Albert Nijenhuis and Herbert S. Wilf, *Combinatorial Algorithms* (second edition), Academic Press, New York, 1978, pp. 26–38.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; and the proposer.

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1487. *Proposed by Kee-Wai Lau, Hong Kong.*

Prove the inequality

$$x + \sin x \geq 2\log(1+x)$$

for $x > -1$.

Combined solutions of Richard I. Hess, Rancho Palos Verdes, California, and the proposer.

Let

$$f(x) = x + \sin x - 2\log(1+x) \quad , \quad x > -1.$$

We have

$$x + \sin x = 2x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = 2x + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

and

$$2\log(1+x) = 2x - \frac{2x^2}{2} + \frac{2x^3}{3} - \frac{2x^4}{4} + \cdots = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1} x^k}{k},$$

and thus

$$\begin{aligned} f(x) &= x^2(1 - \frac{2x}{3} - \frac{x}{3!}) + \frac{x^4}{2}(1 - \frac{4x}{5} + \frac{2x}{5!}) + \frac{x^6}{3}(1 - \frac{6x}{7} - \frac{3x}{7!}) + \cdots \\ &= \sum_{k=1}^{\infty} \frac{x^{2k}}{k} \left(1 - \frac{2k}{2k+1}x + \frac{(-1)^k k}{(2k+1)!}x\right). \end{aligned}$$

Case (i): $-1 < x \leq 1$. Then for each $k \geq 1$,

$$1 - \frac{2k}{2k+1}x + \frac{(-1)^k k}{(2k+1)!}x > 1 - \frac{2k}{2k+1} - \frac{k}{(2k+1)!} = \frac{(2k)! - k}{(2k+1)!} > 0,$$

so $f(x) \geq 0$.

Case (ii): $x \geq 4.5$. Since $x - 2\log(1+x)$ increases for $x > 1$, and

$$f(4.5) \geq 4.5 - 1 - 2\log 5.5 \approx 0.0905038,$$

$f(x) > 0$ for $x \geq 4.5$.

Case (iii): $1 < x < 4.5$. The functions

$$g(x) = x + \sin x \quad \text{and} \quad h(x) = 2\log(1+x)$$

are both nondecreasing, and by means of a calculator we check that

$$g(a) - h(a + 0.05) > 0$$

for $a = 1, 1.05, 1.1, 1.15, \ldots, 4.5$. [*Editor's note.* He's right! In fact the smallest value for $g(a) - h(a+0.05)$ you get is $g(4.05) - h(4.1) \approx 0.0029937$.] Hence for $x \in (a, a+0.05]$ where a is any of the above values,

$$f(x) = g(x) - h(x) \geq g(a) - h(a + 0.05) > 0.$$

Thus $f(x) > 0$ on $(1, 4.5)$ as well.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. There were two incorrect solutions sent in.

Equality holds at $x = 0$, and nearly *does again at around $x = 4$; both Hess and Janous found a relative minimum for the function f at $x \approx 4.06268, f(x) \approx 0.022628$. All three correct proofs had difficulty getting past this point.*

* * * * *

1488. *Proposed by Avinoam Freedman, Teaneck, New Jersey.*

Prove that in any acute triangle, the sum of the circumradius and the inradius is less than the length of the second-longest side.

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

In the figure, O is the circumcenter of a triangle ABC with circumradius R and inradius r, and OL, OM, ON are the perpendiculars to BC, CA, AB, respectively. Then we have

$$R + r = \overline{OL} + \overline{OM} + \overline{ON}$$

(Court, *College Geometry*, p. 73, Theorem 114). We put $\overline{BC} = a, \overline{CA} = b, \overline{AB} = c$, and assume without loss of generality that $a \geq b \geq c$. Let CH be the altitude and S the area of ΔABC. Then

$$\begin{aligned} c \cdot \overline{CH} \;=\; 2S &= a \cdot \overline{OL} + b \cdot \overline{OM} + c \cdot \overline{ON} \\ &\geq c(\overline{OL} + \overline{OM} + \overline{ON}), \end{aligned}$$

so we have (since $\angle A$ is acute)

$$b > \overline{CH} \geq \overline{OL} + \overline{OM} + \overline{ON} = R + r.$$

II. *Solution by the proposer.*

Let the triangle be ABC with $a \leq b \leq c$. Since $90^\circ > B > 90^\circ - A$, we have $\cos B < \cos(90^\circ - A) = \sin A$, and similarly $\cos A < \sin B$. Therefore

$$(1 - \cos A)(1 - \cos B) > (1 - \sin A)(1 - \sin B).$$

Expanding and rearranging, we find

$$\begin{aligned} \sin A + \sin B &> \cos A + \cos B + (\sin A \sin B - \cos A \cos B) \\ &= \cos A + \cos B + \cos C. \end{aligned}$$

Since

$$\sin A = \frac{a}{2R}, \text{etc.,} \quad \text{and} \quad \sum \cos A = 1 + \frac{r}{R} = \frac{R + r}{R},$$

where R is the circumradius and r the inradius, we see that

$$R + r < \frac{a+b}{2} \le b.$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; DUANE M. BROLINE, Eastern Illinois University, Charleston; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; JACK GARFUNKEL, Flushing, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; and BOB PRIELIPP, University of Wisconsin–Oshkosh. Two incorrect solutions were sent in.

The solutions of Bellot Rosado and of Prielipp were similar to but shorter than Solution I, appealing to item 11.16 *of Bottema et al,* Geometric Inequalities *for the inequality* $\overline{CH} \ge R + r$.

* * * * *

1489. *Proposed by M. Selby, University of Windsor.*

Let

$$A_n = (7 + 4\sqrt{3})^n,$$

where n is a positive integer. Find a simple expression for $1 + [A_n] - A_n$, where $[x]$ is the greatest integer less than or equal to x.

Solution by Guo-Gang Gao, student, Université de Montréal.

If

$$(2 + \sqrt{3})^{2n} = a_n + b_n\sqrt{3},$$

where a_n and b_n are integers, then

$$(2 - \sqrt{3})^{2n} = a_n - b_n\sqrt{3},$$

which results from the binomial formula. Therefore

$$(2 + \sqrt{3})^{2n} + (2 - \sqrt{3})^{2n}$$

is an integer. Since $(2 - \sqrt{3})^{2n} < 1$, it follows that

$$[(2 + \sqrt{3})^{2n}] = (2 + \sqrt{3})^{2n} + (2 - \sqrt{3})^{2n} - 1.$$

The above equation can be rewritten as

$$[(7 + 4\sqrt{3})^n] = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n - 1.$$

Therefore,

$$1 + [A_n] - A_n = 1 + [(7 + 4\sqrt{3})^n] - (7 + 4\sqrt{3})^n = (2 - \sqrt{3})^{2n}.$$

Also solved by HAYO AHLBURG, Benidorm, Spain; CURTIS COOPER, Central Missouri State University; NICOS D. DIAMANTIS, student, University of Patras, Greece;

HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; JILL HOUGHTON, Sydney, Australia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; DAVID E. MANES, SUNY at Oneonta, New York; G. NONAY, Wilfrid Laurier University, Waterloo, Ontario; DAVID POOLE, Trent University, Peterborough, Ontario; STANLEY RABINOWITZ, Westford, Massachusetts; CHRIS WILDHAGEN, Breda, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Several solvers gave generalizations.

* * * * *

1490*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

This was suggested by Walther Janous' problem *Crux* 1366 [1989:271]. Find the smallest constant k such that

$$\frac{x}{\sqrt{x+y}}+\frac{y}{\sqrt{y+z}}+\frac{z}{\sqrt{z+x}} \le k\sqrt{x+y+z}$$

for all positive x, y, z.

Solution by G.P. Henderson, Campbellcroft, Ontario.

We will prove that the inequality is true for $k = 5/4$. This is the best possible k because we then have equality for (x, y, z) proportional to $(0, 3, 1)$.

We set

$$x+y=c^2\ ,\ y+z=a^2\ ,\ z+x=b^2,$$

where $a, b, c > 0$ and we assume $a \ge b, c$. Solving these,

$$x=\frac{-a^2+b^2+c^2}{2}\ ,\ y=\frac{a^2-b^2+c^2}{2}\ ,\ z=\frac{a^2+b^2-c^2}{2},$$

and the inequality becomes

$$\frac{-a^2+b^2+c^2}{c}+\frac{a^2-b^2+c^2}{a}+\frac{a^2+b^2-c^2}{b} \le \frac{5}{2\sqrt{2}}\sqrt{a^2+b^2+c^2}. \qquad (1)$$

It is easy to see that

$$\frac{a+\sqrt{b^2+c^2}}{\sqrt{2}} \le \sqrt{a^2+b^2+c^2}$$

[put $\sqrt{b^2+c^2} = d$ and square both sides], so we can replace the right side of (1) by

$$\frac{5}{4}(a+\sqrt{b^2+c^2}).$$

The left side can be written

$$(a+b+c)+\frac{(a+b+c)(a-b)(a-c)(c-b)}{abc}.$$

Since the second part of this just changes sign if we interchange b and c, while the rest of the inequality does not change, we can assume that this term is positive, i.e., $c \geq b$. Multiplying by $4abc$, the inequality to be proved becomes

$$4abc(a+b+c)+4(a+b+c)(a-b)(a-c)(c-b) \leq 5abc(a+\sqrt{b^2+c^2}),$$

or $f(a) \leq 0$, where

$$f(a)=4a^3(c-b)-a^2bc+a(4b^3+4b^2c+4bc^2-4c^3-5bc\sqrt{b^2+c^2})+4bc(c^2-b^2).$$

Since $x \geq 0$, $a \leq \sqrt{b^2+c^2}$ and we are to show that $f(a)$ is negative for

$$b \leq c \leq a \leq \sqrt{b^2+c^2}. \tag{2}$$

If $b=c$,

$$f(a)=-ab^2[(a-b)+(5\sqrt{2}-7)b]<0.$$

If $b<c$, $f(a)$ is a cubic with first and last coefficients greater than 0. We have

$$f(-\infty)<0\ ,\ f(0)>0\ ,\ f(\infty)>0,$$

and we find

$$f(c)=-bc^2(5\sqrt{b^2+c^2}-4b-3c)<0$$

because

$$25(b^2+c^2)-(4b+3c)^2=(3b-4c)^2>0;$$

further,

$$\begin{aligned} f(\sqrt{b^2+c^2}) &= 2bc(4b\sqrt{b^2+c^2}-5b^2-c^2) \\ &= -2bc(\sqrt{b^2+c^2}-2b)^2 \leq 0. \end{aligned}$$

We see that f has three real zeros. One is negative, one is between 0 and c, and one is equal to or greater than $\sqrt{b^2+c^2}$. Therefore f does not change sign in $c \leq a \leq \sqrt{b^2+c^2}$ and is negative for the whole interval, except possibly at $\sqrt{b^2+c^2}$.

The only solution of $f(a)=0$ that satisfies (2) is $a=\sqrt{b^2+c^2}$, and then only if $\sqrt{b^2+c^2}=2b$; that is, when (a,b,c) are proportional to $(2,1,\sqrt{3})$ and so (x,y,z) are proportional to $(0,3,1)$.

Also solved (obtaining the same value $k=5/4$ via a somewhat longer argument) by MARCIN E. KUCZMA, Warszawa, Poland.

At the end of his proof, Kuczma uses the same substitution

$$y+z=a^2\ ,\ z+x=b^2\ ,\ x+y=c^2$$

as Henderson to rewrite the inequality in the form (1)*. He then observes that since the numerators on the left side of* (1) *are all positive,* a, b, c *are the sides of an acute triangle* ABC*. Letting* AP, BQ, CR *be the altitudes, he obtains*

$$\frac{a^2+b^2-c^2}{2b} = \frac{2ab\cos C}{2b} = a\cos C = CQ \text{ , etc.,}$$

and thus the inequality takes the form

$$AR + BP + CQ \le \frac{5}{4\sqrt{2}}\sqrt{a^2+b^2+c^2},$$

with equality for the 30°*–* 60°*–* 90° *triangle. (Not exactly acute, as Kuczma notes!)*

A lovely problem! It seems almost ungrateful to ask if there is a generalization to n *variables . . . .*

* * * * *

1491. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

In triangle ABC, the internal bisector of $\angle A$ meets BC at D, and the external bisectors of $\angle B$ and $\angle C$ meet AC and AB (produced) at E and F respectively. Suppose that the normals to BC, AC, AB at D, E, F, respectively, meet. Prove that $\overline{AB} = \overline{AC}$.

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Because the normals to BC, AC, AB at D, E, F are concurrent (at Q, say), we have

$$\begin{aligned} \overline{BD}^2 + \overline{DQ}^2 &= \overline{BF}^2 + \overline{FQ}^2, \\ \overline{CE}^2 + \overline{EQ}^2 &= \overline{CD}^2 + \overline{DQ}^2, \\ \overline{AF}^2 + \overline{FQ}^2 &= \overline{AE}^2 + \overline{EQ}^2, \end{aligned}$$

so, adding,

$$(\overline{BD}^2 - \overline{DC}^2) + (\overline{CE}^2 - \overline{EA}^2) + (\overline{AF}^2 - \overline{FB}^2) = 0. \quad (1)$$

We put $\overline{BC} = a, \overline{CA} = b, \overline{AB} = c$. In the figure we are assuming $a < b, c$. As AD is the bisector of $\angle A$, we get $\overline{BD} : \overline{DC} = c : b$, so we have

$$\overline{BD} = \frac{ac}{b+c} \text{ , } \overline{DC} = \frac{ab}{b+c}.$$

Similarly $\overline{AE} : \overline{EC} = c : a$ and $\overline{AF} : \overline{FB} = b : a$, so we get

$$\overline{CE} = \frac{ab}{c-a} \text{ , } \overline{EA} = \frac{bc}{c-a},$$

$$\overline{BF} = \frac{ac}{b-a} \text{ , } \overline{AF} = \frac{bc}{b-a}.$$

Therefore from (1) we get

$$\frac{a^2(c^2-b^2)}{(b+c)^2}+\frac{b^2(a^2-c^2)}{(c-a)^2}+\frac{c^2(b^2-a^2)}{(b-a)^2}=0,$$

or

$$\frac{a^2(c-b)}{b+c}-\frac{b^2(a+c)}{c-a}+\frac{c^2(b+a)}{b-a}=0. \qquad (2)$$

In the case $a>b,c$, or $b>a>c$, or $b<a<c$, we have (2) similarly. [Because the above formulas for $\overline{CE}$, etc. will change only in sign.–*Ed.*] The left side of (2) becomes

$$\frac{(c-b)(a+b)(a+c)(b+c-a)^2}{(b+c)(c-a)(b-a)}.$$

[*Editor's note.* Seimiya gave an algebraic derivation. Can anyone find a slick reason why the left side of (2) factors so conveniently?] Because $a,b,c>0$ and $b+c>a$, we obtain $c-b=0$. This implies $\overline{AB}=\overline{AC}$.

II. *Solution by R.H. Eddy, Memorial University of Newfoundland.*

More generally, let the lines AD, BE, CF intersect at a point P with trilinear coordinates (x,y,z) with respect to a given reference triangle ABC with sides a,b,c. If we denote lines through D,E,F by d,e,f and assume that these pass through $Q(u,v,w)$, then it is easy to see that the coordinates of d,e,f are

$$[vz-wy,-uz,uy]\ ,\ [vz,-uz+wx,-vx]\ ,\ [-wy,wx,uy-vx],$$

respectively. Since the condition that lines $[l_1,m_1,n_1]$ and $[l_2,m_2,n_2]$ are perpendicular is

$$l_1l_2+m_1m_2+n_1n_2-(m_1n_2+m_2n_1)\cos A-(n_1l_2+n_2l_1)\cos B-(l_1m_2+l_2m_1)\cos C=0,$$

([1], p. 186), we may write

$$\begin{aligned} d\perp a &\ \Rightarrow\ (z\cos C-y\cos B)u+zv-yw=0,\\ e\perp b &\ \Rightarrow\ zu+(z\cos C-x\cos A)v-xw=0,\\ f\perp c &\ \Rightarrow\ yu-xv+(y\cos B-x\cos A)w=0. \end{aligned}$$

Now, in order for Q to exist, the determinant of the coefficients of u,v,w in this system must vanish, i.e.,

$$\begin{aligned}(y\cos B-z\cos C)(z\cos C-x\cos A)(x\cos A-y\cos B)&\\ +x(z^2-y^2)\cos A+y(x^2-z^2)\cos B+z(y^2-x^2)\cos C&=0. \qquad (3)\end{aligned}$$

If $P=I(1,1,1)$, the incentre of ABC, then (3) becomes

$$(\cos B-\cos C)(\cos C-\cos A)(\cos A-\cos B)=0,$$

i.e., ABC must be isosceles in one of the three ways. In the proposal, $P=I_A(-1,1,1)$, the excentre opposite A, which implies $B=C$ as required.

One can check that the coordinates of the centroid (bc, ca, ab) and the Gergonne point

$$\left(\frac{1}{a(s-a)}\ ,\ \frac{1}{b(s-b)}\ ,\ \frac{1}{c(s-c)}\right)$$

(s the semiperimeter) also satisfy the above system, in which case Q is the circumcentre and the incentre respectively. Are there any other interesting pairs?

Reference:

[1] D.M.Y. Sommerville, *Analytical Conics*, G. Bell and Sons, Ltd., London, 1961.

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; MARIA ASCENSIÓN LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; and the proposer.

The proposer mentioned that the analogous problem for three interior bisectors was solved by Thébault. This case is contained in solution II *above.*

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