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CONTENTS

TWO THEOREMS ON TANGENTS TO A PARABOLA

Dan Pedoe

Let points $P_0, P_1, P_2, \ldots$ on a line p be related by a similarity of the Euclidean plane to points $Q_0, Q_1, Q_2, \ldots$ on a line q, that is,

$$P_iP_j : P_jP_k = Q_iQ_j : Q_jQ_k$$

for all $i \neq j \neq k$, and suppose $Q_0 = P_{n_0}$ is the point of intersection of p and q, where $n_0 \neq 0$. Then our first theorem was the subject of an earlier article [2] in *Crux*:

Theorem I. The joins P_iQ_i $(i = 0, 1, 2, \ldots)$ envelope a parabola, which also touches p and q.

Proof. We use some notions of projective geometry (see [1], p. 328): $P_iP_j : P_jP_k$ is the cross-ratio $\{P_iP_k, P_j\infty\}$, so that $\{P_iP_k, P_j\infty\} = \{Q_iQ_k, Q_j\infty\}$, and the range (P_i) is projective with the range (Q_i). Hence since $P_0 \neq Q_0$ ([1], p. 330) the lines P_iQ_i touch a conic. Among these lines is the line $\infty\infty$, which denotes the line at infinity, so that the conic is a parabola. Also P_0Q_0 is the line p, and $P_{n_0}Q_{n_0}$ is the line q, and p and q also touch the parabola. □

This envelope appears in kindergarten drawings, and is evidently ornamental, but it is sometimes thought, incorrectly, to be the envelope of an hyperbola, or even a circle (see [2]).

Our second theorem turned up more recently in a solution of *Crux* 1388 [1990: 24], and appears to be distinct from the first, but the two are closely related. It is worth remembering that affine transformations do not change ratios of segments on the same line, or on parallel lines, and that under affine transformations there is essentially only one parabola.

Theorem II. A given triangle $T_1T_2T_3$ has sides t_1, t_2, t_3, and is intersected by a line t in the points Q_1, Q_2, Q_3 in order, where $Q_2Q_3 = kQ_1Q_2$ for *fixed* k. Then the lines t envelope a parabola, which touches t_1, t_2 and t_3.

Proof. Let t_4 be one position of the line t. Since a conic is uniquely determined if we are given five tangents, and the line at infinity touches a parabola, there is a unique parabola P which touches t_1, t_2, t_3 and t_4. Let P_1, P_2 and P_3 be the respective points of contact of t_1, t_2 and t_3 with the parabola P. Then if t is any tangent to P, and the respective intersections of t with t_1, t_2 and t_3 are R_1, R_2 and R_3, by [1], p. 330, Theorem II,

$$\{Q_1Q_3, Q_2\infty\} = \{P_1P_3, P_2\infty\} = \{R_1R_3, R_2\infty\},$$

where the first and third cross-ratios are taken on their respective lines, but the second cross-ratio is taken on the parabola P: that is, it is the cross-ratio of the pencil

$V(P_1P_3, P_2\infty)$, where V is any point on the parabola P. Hence $Q_1Q_2 : Q_2Q_3 = R_1R_2 : R_2R_3$, and all tangents t to the parabola P intersect the sides of $T_1T_2T_3$ in points R_1, R_2, R_3 with $R_2R_3 = kR_1R_2$.

There are no other lines which do this, since a simple *reductio ad absurdum* shows that through a given point R_1 of t_1 there is only one line which intersects t_2 in R_2 and t_3 in R_3 with $R_2R_3 = kR_1R_2$, and of course there is only one tangent to P from a point on t_1, this line being itself a tangent. $\square$

The connection with Theorem I is clear if we consider a tangent t_5 to P distinct from t_4 and the sides of triangle $T_1T_2T_3$, and observe that tangents to the parabola intersect t_4 and t_5 in similar ranges. If t'_1, t'_2, t'_3 are three such tangents, and Q'_1, Q'_2, Q'_3 are the intersections with t_4, and R'_1, R'_2, R'_3 the intersections with t_5, then $Q'_2Q'_3 = k'Q'_1Q'_2$ and $R'_2R'_3 = k'R'_1R'_2$, where the value of k' depends on the points of contact of t'_1, t'_2 and t'_3 with the parabola P.

Both theorems, of course, are embraced by the theorem that two given tangents to a parabola are cut in projective ranges by the other tangents to the parabola, and the fact that the points at infinity on each of the given tangents correspond. If we select three points on one given tangent, and the corresponding three points on the other tangent, we are back in the situations considered above.

As a final remark, suppose that we wished to illustrate the case $k = 1$: that is, given a parabola P, find a triangle $T_1T_2T_3$ of tangents such that all other tangents intersect the sides of $T_1T_2T_3$ in points Q_1, Q_2 and Q_3 with $Q_1Q_2 = Q_2Q_3$. (This was the case used in the proof of *Crux* 1388 [1990: 24].) If the points of contact of the sides of triangle $T_1T_2T_3$ with the given parabola are P_1, P_2 and P_3, then $\{P_1P_3, P_2\infty\} = -1$, so that the lines P_1P_3 and $P_2\infty$ are conjugate. Let P_1 and P_3 be two points of P and define T_2 to be the point of intersection of the tangents to P at P_1 and P_3; draw the line through T_2 and the point at infinity of the parabola (i.e., parallel to the axis of the parabola). If this intersects the parabola at the point P_2, then the tangent to the parabola at P_2 is the side T_1T_3 we are looking for.

References:
[1] Dan Pedoe, *Geometry: A Comprehensive Course*, Dover Publications, 1988.
[2] Dan Pedoe, A parabola is not an hyperbola, *Crux Math.* **5** (1979) 122–124.

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THE OLYMPIAD CORNER

No. 124

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first problem set we give is the 9th annual American Invitational Mathematics Examination (A.I.M.E.) written Tuesday, March 19, 1991. The time allowed was 3 hours. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. Only the numerical solutions will be published next month. Full solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A., 68588-0322.

1991 AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

1. Find x^2+y^2 if x and y are positive integers such that

$$xy+x+y=71 \qquad \text{and} \qquad x^2y+xy^2=880.$$

2. Rectangle $ABCD$ has sides $\overline{AB}$ of length 4 and $\overline{CB}$ of length 3. Divide $\overline{AB}$ into 168 congruent segments with points $A=P_0, P_1, \ldots, P_{168}=B$, and divide $\overline{CB}$ into 168 congruent segments with points $C=Q_0, Q_1, \ldots, Q_{168}=B$. For $1 \le k \le 167$, draw the segments $\overline{P_kQ_k}$. Repeat this construction on the sides $\overline{AD}$ and $\overline{CD}$, and then draw the diagonal $\overline{AC}$. Find the sum of the lengths of the 335 parallel segments drawn.

3. Expanding $(1+0.2)^{1000}$ by the binomial theorem and doing no further manipulation gives

$$\binom{1000}{0}(0.2)^0+\binom{1000}{1}(0.2)^1+\binom{1000}{2}(0.2)^2+\cdots+\binom{1000}{1000}(0.2)^{1000}=A_0+A_1+\cdots+A_{1000},$$

where $A_k=\binom{1000}{k}(0.2)^k$ for $k=0,1,2,\ldots,1000$. For which k is A_k the largest?

4. How many real numbers x satisfy the equation $\frac{1}{5}\log_2 x=\sin(5\pi x)$?

5. Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many rational numbers between 0 and 1 will 20! be the resulting product?

6. Suppose r is a real number for which

$$\left\lfloor r+\frac{19}{100}\right\rfloor+\left\lfloor r+\frac{20}{100}\right\rfloor+\left\lfloor r+\frac{21}{100}\right\rfloor+\cdots+\left\lfloor r+\frac{91}{100}\right\rfloor=546.$$

Find $\lfloor 100r \rfloor$. (For real x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

7. Find A^2, where A is the sum of the absolute values of all roots of the following equation:

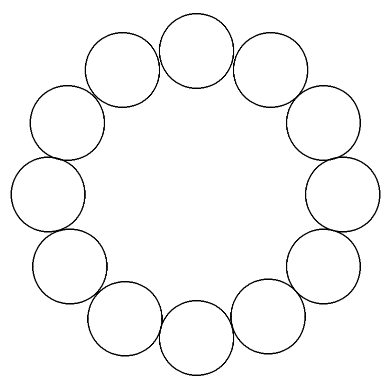
$$x = \sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{x}}}}}$$

8. For how many real numbers a does the quadratic equation $x^2 + ax + 6a = 0$ have only integer roots for x?

9. Suppose that $\sec x + \tan x = \frac{22}{7}$ and that $\csc x + \cot x = \frac{m}{n}$, where $\frac{m}{n}$ is in lowest terms. Find $m + n$.

10. Two three-letter strings, aaa and bbb, are transmitted electronically. Each string is sent letter by letter. Due to faulty equipment, each of the six letters has a 1/3 chance of being received incorrectly, as an a when it should have been a b, or as a b when it should have been an a. However, whether a given letter is received correctly or incorrectly is independent of the reception of any other letter. Let S_a be the three-letter string received when aaa is transmitted and let S_b be the three-letter string received when bbb is transmitted. Let p be the probability that S_a comes before S_b in alphabetical order. When p is written as a fraction in lowest terms, what is its numerator?

11. Twelve congruent disks are placed on a circle C of radius 1 in such a way that the twelve disks cover C, no two of the disks overlap, and so that each of the twelve disks is tangent to its two neighbors. The resulting arrangement of disks is shown in the figure to the right. The sum of the areas of the twelve disks can be written in the form $\pi(a \Leftrightarrow b\sqrt{c})$, where a, b, c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.



12. Rhombus $PQRS$ is inscribed in rectangle $ABCD$ so that vertices P, Q, R, and S are interior points on sides $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$, respectively. It is given that $PB = 15$, $BQ = 20$, $PR = 30$, and $QS = 40$. Let m/n, in lowest terms, denote the perimeter of $ABCD$. Find $m + n$.

13. A drawer contains a mixture of red socks and blue socks, at most 1991 in all. It so happens that, when two socks are selected randomly without replacement, there is a probability of exactly 1/2 that both are red or both are blue. What is the largest possible number of red socks in the drawer that is consistent with this data?

14. A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.

15. For positive integer n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n} \sqrt{(2k \Leftrightarrow 1)^2 + a_k^2},$$

where $a_1, a_2, \ldots, a_n$ are positive real numbers whose sum is 17. There is a unique positive integer n for which S_n is also an integer. Find this n.

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This month we also give the 1990 Australian Olympiad. I particularly want to thank Andy Liu, University of Alberta, for having collected these, and other problem sets we shall use, while he was at the IMO last summer.

1990 AUSTRALIAN MATHEMATICAL OLYMPIAD

Paper I: Tuesday, 13th February, 1990

Time allowed: 4 hours

1. Let f be a function defined for all real numbers and taking real numbers as its values. Suppose that, for all real numbers x, y, the function f satisfies

$$(1)\ f(2x) = f\left(\sin\left(\frac{\pi x}{2} + \frac{\pi y}{2}\right)\right) + f\left(\sin\left(\frac{\pi x}{2} \Leftrightarrow \frac{\pi y}{2}\right)\right),$$

$$(2)\ f(x^2 \Leftrightarrow y^2) = (x+y)f(x \Leftrightarrow y) + (x \Leftrightarrow y)f(x+y).$$

Show that these conditions uniquely determine

$$f(1990 + 1990^{1/2} + 1990^{1/3})$$

and give its value.

2. Prove that there are infinitely many pairs of positive integers m and n such that n is a factor of $m^2 + 1$ and m is a factor of $n^2 + 1$.

3. Let ABC be a triangle and k_1 be a circle through the points A and C such that k_1 intersects AB and BC a second time in the points K and N respectively, K and N being different. Let O be the centre of k_1. Let k_2 be the circumcircle of the triangle KBN, and let the circumcircle of the triangle ABC intersect k_2 also in M, a point different from B. Prove that OM and MB are perpendicular.

4. A solitaire game is played with an even number of discs, each coloured red on one side and green on the other side. Each disc is also numbered, and there are two of each number; i.e. $\{1, 1, 2, 2, 3, 3, \ldots, N, N\}$ are the labels. The discs are laid out in rows with each row having at least three discs. A move in this game consists of flipping over simultaneously two discs with the same label. Prove that for *every* initial deal or layout there is a sequence of moves that ends with a position in which no row has only red or only green sides showing.

Paper II: Wednesday, 14th February, 1990

Time allowed: 4 hours.

5. In a given plane, let K and k be circles with radii R and r, respectively, and suppose that K and k intersect in precisely two points S and T. Let the tangent to k through S intersect K also in B, and suppose that B lies on the common tangent to k and K. Prove: if ϕ is the (interior) angle between the tangents of K and k at S, then

$$\frac{r}{R} = (2\sin\frac{\phi}{2})^2.$$

6. Up until now the National Library of the small city state of Sepharia has had n shelves, each shelf carrying at least one book. The library recently bought k new shelves, k being positive. The books will be rearranged, and the librarian has announced that each of the now $n+k$ shelves will contain at least one book. Call a book *privileged* if the shelf on which it will stand in the new arrangement is to carry fewer books than the shelf which has carried it so far. Prove: there are at least $k+1$ privileged books in the National Library of Sepharia.

7. For each positive integer n, let $d(n)$ be the number of distinct positive integers that divide n. Determine all positive integers for which $d(n) = n/3$ holds.

8. Let n be a positive integer. Prove that

$$\frac{1}{\binom{2n}{1}} \Leftrightarrow \frac{1}{\binom{2n}{2}} + \frac{1}{\binom{2n}{3}} \Leftrightarrow \cdots + \frac{(\Leftrightarrow 1)^{k-1}}{\binom{2n}{k}} + \cdots + \frac{1}{\binom{2n}{2n-1}} = \frac{1}{n+1}.$$

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Before turning to solutions submitted by the readers, I want to give two comments received about recent numbers of the Corner.

4. [1985: 304; 1991: 41] *Proposed by Canada.*

Prove that

$$\frac{x_1^2}{x_1^2 + x_2x_3} + \frac{x_2^2}{x_2^2 + x_3x_4} + \cdots + \frac{x_n^2}{x_n^2 + x_1x_2} \le n \Leftrightarrow 1$$

where all $x_i > 0$.

Comment by Murray S. Klamkin, University of Alberta.

A more general version of this problem, with solution, appears as *Crux* 1429 [1990: 155].

1. [1982: 237; 1990: 227] *31st Bulgarian Mathematical Olympiad.*

Find all pairs of natural numbers (n, k) for which $(n+1)^k \Leftrightarrow 1 = n!$.

Editor's note. In the October 1990 issue of *Crux* I discussed a solution to this problem, saying it had not been considered in the Corner. Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario, takes me to task for not remembering what I have published in earlier numbers! He points out that this problem was also given as number 1 of the 1984 Brazilian Mathematical Olympiad [1987: 70] and that in [1988: 231] I had given George Evagelopoulos's observation that it was also problem 4 of the 1983 Australian Olympiad, and its solution is in [1986: 23]. So the same problem was used at least three times in three different Mathematical Olympiads, and in three consecutive years! Wang asks if this is a record.

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We next give some solutions to "archive problems". I hope they haven't been discussed before!

2. [1981: 47] *1978 Romanian Mathematical Olympiad, Final Round (12th class).*

Let P and Q be two polynomials (neither identically zero) with complex coefficients. Show that P and Q have the same roots (with the same multiplicities) if and only if the function $f : \mathbf{C} \to \mathbf{R}$ defined by $f(z) = |P(z)| - |Q(z)|$ has a constant sign for all $z \in C$ if it is not identically zero.

Solution by Murray S. Klamkin, University of Alberta.

The *only if* part is easy.

For the *if* part, we can assume without loss of generality that $f(z) \geq 0$. If r is any root of P, it immediately follows that it must also be a root of Q (note if P is a constant, then so also Q is a constant). Also the multiplicity of any root of P must be at most the corresponding multiplicity in Q. For if a root r had greater multiplicity in P than in Q, by setting $z = r + \varepsilon$, where ε is arbitrarily small, we would have $f(z) < 0$. Next the degree of P must be at least the degree of Q. Otherwise by taking $|z|$ arbitrarily large, we would have $f(z) < 0$. It follows that P and Q have the same roots with the same multiplicities.

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4. [1982: 301] *U.S. Olympiad Student Proposals.*

Find all solutions (x, y, z) of the Diophantine equation

$$x^3 + y^3 + z^3 + 6xyz = 0.$$

Comment by Murray S. Klamkin, University of Alberta.

The only nonzero solutions are $(x, y, z) = (1, -1, 0)$ in some order.

Mordell [1] has shown that the equation $x^3 + y^3 + z^3 + dxyz = 0$, for $d \neq 1, -3, -5$, has either three relatively prime solutions or an infinite number of solutions. Three of the solutions are $(1, -1, 0), (0, 1, -1)$ and $(-1, 0, 1)$. (Note that, e.g., $(1, -1, 0)$ and $(-1, 1, 0)$ are considered the same.) According to Dickson [2], Sylvester stated that $F \equiv x^3 + y^3 + z^3 + 6xyz$ is not solvable in integers (presumably non-trivially). The same holds for $2F = 27nxyz$ when $27n^2 - 8n + 4$ is a prime, and for $4F = 27nxyz$ when $27n^2 - 36n + 16$ is a prime.

References:
[1] L.J. Mordell, *Diophantine Equations*, Academic Press, New York, 1969, p. 78.
[2] L.E. Dickson, *History of the Theory of Numbers, Vol. II*, Stechert, New York, 1939, p. 589.

7. [1982: 301] *U.S. Olympiad Student Proposals.*
In $\mathbf{R}^n$ let $\mathbf{X} = (x_1, x_2, \ldots, x_n)$, $\mathbf{Y} = (y_1, y_2, \ldots, y_n)$, and, for $p \in (0,1)$, define

$$F_p(\mathbf{X},\mathbf{Y}) \equiv \left(\left|\frac{x_1}{p}\right|^p \left|\frac{y_1}{1 \Leftrightarrow p}\right|^{1-p}, \ \left|\frac{x_2}{p}\right|^p \left|\frac{y_2}{1 \Leftrightarrow p}\right|^{1-p}, \ \ldots, \ \left|\frac{x_n}{p}\right|^p \left|\frac{y_n}{1 \Leftrightarrow p}\right|^{1-p} \right).$$

Prove that

$$\|\mathbf{X}\|_m + \|\mathbf{Y}\|_m \geq \|F_p(\mathbf{X},\mathbf{Y})\|_m,$$

where

$$\|\mathbf{X}\|_m = (|x_1|^m + |x_2|^m + \cdots + |x_n|^m)^{1/m}.$$

Solution by Murray S. Klamkin, University of Alberta.
By Hölder's inequality

$$\|F_p(\mathbf{X},\mathbf{Y})\| \leq \frac{\|\mathbf{X}\|_m^p \ \|\mathbf{Y}\|_m^{1-p}}{p^p(1 \Leftrightarrow p)^{1-p}}.$$

By the weighted A.M.-G.M. inequality,

$$\|\mathbf{X}\|_m + \|\mathbf{Y}\|_m = p\left\|\frac{\mathbf{X}}{p}\right\|_m + \ (1 \Leftrightarrow p)\left\|\frac{\mathbf{Y}}{1 \Leftrightarrow p}\right\|_m \geq \frac{\|\mathbf{X}\|_m^p \ \|\mathbf{Y}\|_m^{1-p}}{p^p(1 \Leftrightarrow p)^{1-p}}.$$

The result follows.

If $m \geq 1$, and $x_i, y_i > 0$ for all i, then another proof, via Minkowski's inequality and the weighted A.M.-G.M. inequality, is

$$\|\mathbf{X}\|_m + \|\mathbf{Y}\|_m \geq \|\mathbf{X} + \mathbf{Y}\|_m \geq \|F_p(\mathbf{X},\mathbf{Y})\|_m.$$

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3. [1983: 107] *1983 Bulgarian Winter Competition.*
Determine all values of the real parameter p for which the system of equations

$$x + y + z = 2$$

$$yz + zx + xy = 1$$

$$xyz = p$$

has a real solution.

Solution by Murray S. Klamkin, University of Alberta.

Since

$$x = xyz + x^2z + x^2y = p + x^2(2 \Leftrightarrow x) = p + 2x^2 \Leftrightarrow x^3,$$

the solution (x, y, z) is given by the three roots, in any order, of the cubic equation

$$t^3 \Leftrightarrow 2t^2 + t \Leftrightarrow p = 0.$$

As is known [1], the condition that the general cubic equation

$$at^3 + 3bt^2 + 3ct + d = 0$$

have real roots is that $\Delta \leq 0$ where

$$\Delta = a^2d^2 \Leftrightarrow 6abcd + 4ac^3 + 4db^3 \Leftrightarrow 3b^2c^2.$$

For the case here,

$$\Delta = p^2 \Leftrightarrow 4p/27.$$

Consequently, p must lie in the closed interval [0,4/27].

References:
[1] S. Barnard, J.M. Child, *Higher Algebra*, MacMillan, London, 1949, pp. 179-180.

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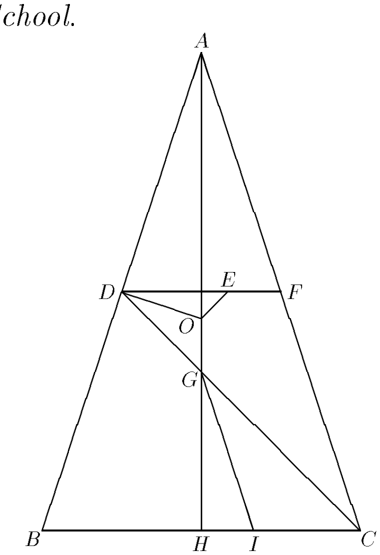
1. [1983: 108] *1983 British Mathematical Olympiad.*

In the triangle ABC with circumcentre O , $AB = AC$, D is the midpoint of AB, and E is the centroid of triangle ACD. Prove that OE is perpendicular to CD.

Solution by Jie Lou, student, Halifax West High School.



Join the lines DE, DO, and AO, and let F be the intersection of DE and AC, G the intersection of AO and CD, and H the intersection of AO and BC. Find the point I on BC such that $HI = \frac{1}{3}HC$. Since ΔABC is isosceles and O is the circumcentre, AO is the central line of BC. Since D is the midpoint of AB, G is the centroid of the triangle, and $GH = \frac{1}{3}AH$. Thus $GI \| AC$. Therefore $\angle HGI = \angle HAC = \angle DAO$. Since O is the circumcentre and D is the midpoint of AB, OD is perpendicular to AB. Also, we have $\angle GHI = 90°$. Then $\Delta GHI \sim \Delta ADO$. From this we have $GH/AD = HI/DO$. Now, since $DE = \frac{2}{3}DF = \frac{2}{3}CH = 2HI$ and $AG = 2GH$, we have

$$\frac{AG}{AD} = \frac{DE}{DO}.$$

Obviously, DE is perpendicular to AH, so that $\angle ODE = 90° \Leftrightarrow \angle ADE = \angle DAG$. From this, $\Delta ADG \sim \Delta DOE$. Since the angle between AD and DO is $90°$, the angle between DG and EO must be $90°$, too. Thus OE is perpendicular to CD.

3. [1983: 108] *1983 British Mathematical Olympiad.*
The real numbers $x_1, x_2, x_3, \ldots$ are defined by

$$x_1 = a \neq -1 \quad \text{and} \quad x_{n+1} = x_n^2 + x_n \text{ for all } n \geq 1.$$

S_n is the sum and P_n is the product of the first n terms of the sequence $y_1, y_2, y_3, \ldots$, where

$$y_n = \frac{1}{1 + x_n}.$$

Prove that $aS_n + P_n = 1$ for all n.

Solution by Murray S. Klamkin, University of Alberta.
It follows easily that

$$\frac{1}{y_{n+1}} = \frac{1}{y_n^2} - \frac{1}{y_n} + 1, \tag{1}$$

where $y_1 = 1/(1+a)$. Now let

$$\varphi_n = aS_{n+1} + P_{n+1} - aS_n - P_n = P_n(y_{n+1} - 1) + ay_{n+1}. \tag{2}$$

Replacing n by $n - 1$ and dividing gives

$$\frac{y_n(y_{n+1} - 1)}{y_n - 1} = \frac{\varphi_n - ay_{n+1}}{\varphi_{n-1} - ay_n}.$$

It follows from (1) that

$$\frac{y_n(y_{n+1} - 1)}{y_n - 1} = \frac{y_{n+1}}{y_n}.$$

Hence

$$\frac{\varphi_n - ay_{n+1}}{\varphi_{n-1} - ay_n} = \frac{y_{n+1}}{y_n}$$

or

$$\varphi_n y_n = \varphi_{n-1} y_{n+1}. \tag{3}$$

An easy calculation shows that $y_2 = 1/(1 + a + a^2)$, so that

$$\begin{aligned} \varphi_1 &= P_1(y_2 - 1) + ay_2 = \frac{1}{1+a}\left(\frac{1}{1+a+a^2} - 1\right) + \frac{a}{1+a+a^2} \\ &= \frac{1 + a(1+a)}{(1+a)(1+a+a^2)} - \frac{1}{1+a} = 0. \end{aligned}$$

Then since $y_i \neq 0$ for all i, from (3) $\varphi_i = 0$ for all i. Since $aS_1 + P_1 = 1$, by (2) $aS_n + P_n = 1$ for all n.

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P375 [1983: 237] *From Középiskolai Matematikai Lapok.*

Does there exist a function $f : \mathbf{R} \to \mathbf{R}$ such that $\lim_{x\to\infty} f(x) = \infty$ and

$$\lim_{x\to\infty} \frac{f(x)}{\ln(\ln(\ldots(\ln x)\ldots))} = 0$$

holds for all n (where n is the number of logarithm functions in the denominator)?

Solution by Murray S. Klamkin, University of Alberta

The answer to the given problem is in the affirmative and it is a special case of a result of du Bois-Reymond [1].

First, if $f(x)/g(x) \to \infty$ as $x \to \infty$, we say that the order of f is greater than the order of g and we write it as $f \succ g$. The theorem of du Bois-Reymond is that given a scale of increasing functions φ_n such that

$$\varphi_1 \succ \varphi_2 \succ \varphi_3 \succ \ldots \succ 1,$$

then there exists an increasing function f such that $\varphi_n \succ f \succ 1$ for all values of n. Here we choose $\varphi_1 = \ln x, \varphi_2 = \ln\ln x, \varphi_3 = \ln\ln\ln x$, etc.

More generally we have the following: given a descending sequence $\{\varphi_n\} : \varphi_1 \succ \varphi_2 \succ \varphi_3 \succ \cdots \succ \varphi_n \succ \cdots \succ \varphi$ and an ascending sequence $\{\psi_n\} : \psi_1 \prec \psi_2 \prec \psi_3 \prec \cdots \prec \psi_p \prec \cdots \prec \psi$ such that $\psi_p \prec \varphi_n$ for all n and p then there is f such that $\psi_p \prec f \prec \varphi_n$ for all n and p.

References:

[1] P. du Bois-Reymond, Über asymptotische Werthe, infinitare Approximationen und infinitare Auflösung von Gleichungen, *Math. Annalen* 8 (1875), p. 365.

*

1. [1984: 74] *West Point Proposals.*

Given six segments $S_1, S_2, \ldots, S_6$ congruent to the edges AB, AC, AD, CD, DB, BC, respectively, of a tetrahedron $ABCD$, show how to construct with straightedge and compass a segment whose length equals that of the *bialtitude* of the tetrahedron relative to opposite edges AB and CD (i.e., the distance between the lines AB and CD).

Solution by Murray S. Klamkin, University of Alberta.

We use the known construction for an altitude of a tetrahedron [1] and the known theorem [2] that the volume of a tetrahedron equals one-sixth the product of two opposite edges times the sine of the angle between those edges and times the shortest distance between those edges. The volume also equals one-third the product of an altitude and the area of the corresponding face.

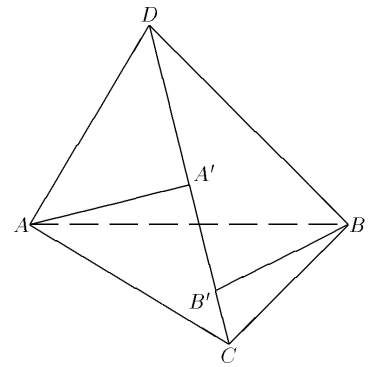
Let $s_1 = |S_1|$, etc. Let θ be the angle between edges AB and CD and let d be the distance between AB and CD. Also let h_D be the altitude of the tetrahedron from D and h' be the altitude of triangle ABC from C. Then six times the volume of the tetrahedron equals

$$s_4 s_1 d \sin\theta = 2h_D[ABC] = h_D s_1 s_2 \sin\angle CAB = h_D s_1 h'. \qquad (1)$$

Here $[ABC]$ denotes the area of triangle ABC. To express $\sin\theta$ as a ratio of two constructible segments, we have

$$s_4 s_1 \cos\theta = |(\mathbf{A} \Leftrightarrow \mathbf{B}) \cdot \mathbf{C}| = |\mathbf{A} \cdot \mathbf{C} \Leftrightarrow \mathbf{B} \cdot \mathbf{C}| = |s_3 s_4 \cos\angle ADC \Leftrightarrow s_5 s_4 \cos\angle BDC|, \quad (2)$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are respective vectors from D to A, B, C. Now if AA' and BB' are the respective altitudes in triangles ACD and BCD, then $s_3 \cos\angle ADC = A'D$ and $s_5 \cos\angle BDC = B'D$. Hence from (2),



$$\cos\theta = \frac{|A'D \Leftrightarrow B'D|}{s_1} = \frac{A'B'}{s_1},$$

where the length $A'B'$ is constructible. From this, we easily can obtain $\sin\theta = \alpha/s_1$ where $\alpha = \sqrt{s_1^2 \Leftrightarrow (A'B')^2}$ is also constructible. Finally from (1),

$$d = \frac{h_D h' s_1}{s_4 \alpha}$$

which is constructible in the usual way.

References:

[1] M.S. Klamkin, *USA Mathematical Olympiads 1972-1986*, Math. Association of America, Washington, D.C., 1988, p. 59.

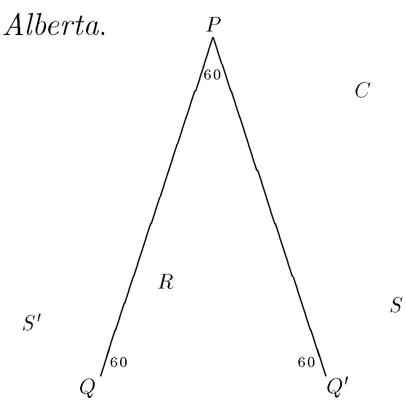
[2] N. Altshiller-Court, *Modern Pure Solid Geometry*, Macmillan, New York, 1935, p. 86, Thm. 275.

2. [1984: 108] *1982 Austrian-Polish Mathematics Competition.*

We are given a unit circle C with center M and a closed convex region R in the interior of C. From every point P of circle C, there are two tangents to the boundary of R that are inclined to each other at $60°$. Prove that R is a closed circular disk with center M and radius $1/2$.

Solution by Murray S. Klamkin, University of Alberta.

Let P be on C and let the two tangents from P to R meet C again at Q and Q'. Let the other tangent from $Q(Q')$ meet the circle again at $S(S')$ respectively. Since angles P, Q, Q' are all $60°$, QS must coincide with $Q'S'$ so that PQS is an equilateral triangle. Hence R is the envelope of all inscribed equilateral triangles in a circle. This envelope is known to be a circle concentric with C and with radius half that of C.



* * *

We finish this month's Corner with solutions to four of the five problems of the *1989 Asian Pacific Mathematical Olympiad* [1989: 131]. We're still waiting for a correct proof of number 2.

1. Let $x_1, x_2, \ldots, x_n$ be positive real numbers, and let $S = x_1 + x_2 + \cdots + x_n$. Prove that

$$(1+x_1)(1+x_2)\cdots(1+x_n) \leq 1 + S + \frac{S}{2!} + \frac{S}{3!} + \cdots + \frac{S}{n!}\ .$$

Correction and solution by George Evagelopoulos, Athens, Greece, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The correct inequality should be

$$(1+x_1)(1+x_2)\cdots(1+x_n) \leq 1 + S + \frac{S^2}{2!} + \cdots + \frac{S^n}{n!}.$$

This is well known and can be found, for example, in *Analytic Inequalities* by D.S. Mitrinović (§§3.2.42). Now by the AM-GM inequality

$$\begin{aligned}(1+x_1)\cdots(1+x_n) &\leq \left(\frac{n + x_1 + \ldots + x_n}{n}\right)^n = \left(1 + \frac{S}{n}\right)^n \\ &= 1 + n\left(\frac{S}{n}\right) + \frac{n(n-1)}{2}\left(\frac{S}{n}\right)^2 + \cdots + \left(\frac{S}{n}\right)^n\end{aligned}$$

using the binomial theorem. Since $(n-m)!n^m \geq n!$, the coefficient of S^m is

$$\binom{n}{m}\frac{1}{n^m} = \frac{n!}{m!(n-m)!n^m} \leq \frac{n!}{m!n!} = \frac{1}{m!},$$

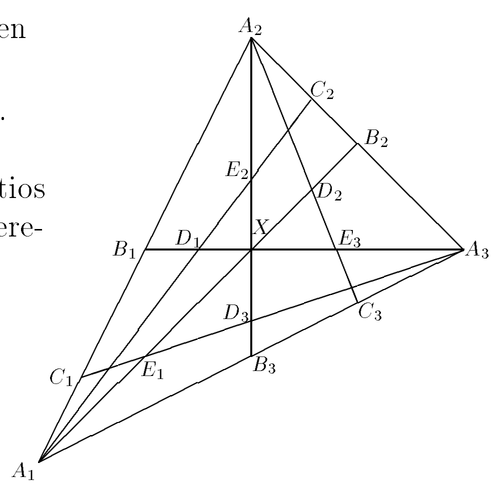
from which the result is immediate.

3. Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1$, $A_5 = A_2$. For $n = 1, 2$ and 3 suppose that B_n is the midpoint of A_nA_{n+1}, and suppose that C_n is the midpoint of A_nB_n. Suppose that A_nC_{n+1} and B_nA_{n+2} meet at D_n, and that A_nB_{n+1} and C_nA_{n+2} meet at E_n. Calculate the ratio of the area of triangle $D_1D_2D_3$ to the area of triangle $E_1E_2E_3$.

Solution by George Evagelopoulos, Athens, Greece.

Let X denote the centroid of $\Delta A_1A_2A_3$. Then

$$B_1X = \frac{1}{3}A_3B_1\ ,\ B_2X = \frac{1}{3}A_1B_2\ ,\ B_3X = \frac{1}{3}A_2B_3.$$



Using perspectives from A_1 the cross-ratios $(A_2, A_3; B_2, C_2)$ and $(B_1, A_3; X, D_1)$ are equal. Therefore

$$\frac{A_2B_2}{B_2A_3}\cdot\frac{C_2A_3}{A_2C_2} = \frac{B_1X}{XA_3}\cdot\frac{D_1A_3}{B_1D_1}$$

from which

$$1 \cdot 3 = \frac{1}{2}\cdot\frac{D_1A_3}{B_1D_1}.$$

This gives $D_1A_3 = 6B_1D_1$ and so $B_1D_1 = \frac{1}{7}B_1A_3$. Hence

$$\frac{D_1X}{B_1X} = \frac{B_1X \Leftrightarrow B_1D_1}{B_1X} = \frac{\frac{1}{3} \Leftrightarrow \frac{1}{7}}{\frac{1}{3}} = \frac{4}{7}.$$

Similarly $D_2X/B_2X = \frac{4}{7}$ and ΔD_1D_2X is similar to ΔB_1B_2X. Thus

$$\frac{[D_1D_2X]}{[B_1B_2X]} = \left(\frac{4}{7}\right)^2 = \frac{16}{49},$$

where $[T]$ denotes the area of triangle T. In similar fashion

$$\frac{[D_2D_3X]}{[B_2B_3X]} = \frac{[D_1D_3X]}{[B_1B_3X]} = \frac{16}{49}.$$

It follows that

$$[D_1D_2D_3] = \frac{16}{49}[B_1B_2B_3] = \frac{4}{49}[A_1A_2A_3],$$

using the fact that triangles $B_1B_2B_3, B_3A_1B_1, B_2B_1A_2$ and $A_3B_3B_2$ are congruent.

Now, with A_2 as the centre

$$(A_1, A_3; B_3, C_3) = (B_1, A_3; X, E_3)$$

and so

$$1 \cdot \frac{1}{3} = \frac{A_1B_3}{B_3A_3} \cdot \frac{C_3A_3}{A_1C_3} = \frac{B_1X}{XA_3} \cdot \frac{E_3A_3}{B_1E_3} = \frac{1}{2} \cdot \frac{E_3A_3}{B_1E_3}.$$

Since $XA_3 = \frac{2}{3}B_1A_3$, this means

$$XE_3 = \left[\frac{2}{3} \Leftrightarrow \frac{2}{5}\right] B_1A_3 = \frac{4}{15}B_1A_3.$$

Hence $XE_3 = \frac{2}{5}XA_3$ and $[XE_3E_1] = \frac{4}{25}[XA_3A_1]$. Finally $[E_1E_2E_3] = \frac{4}{25}[A_1A_2A_3]$ so that

$$[D_1D_2D_3] = \frac{25}{49}[E_1E_2E_3].$$

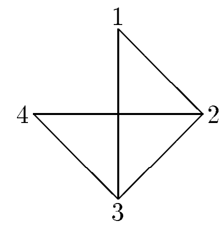
4. Let S be a set consisting of m pairs (a, b) of positive integers with the property that $1 \le a < b \le n$. Show that there are at least

$$4m \cdot \frac{m \Leftrightarrow n^2/4}{3n}$$

triples (a, b, c) such that (a, b), (a, c) and (b, c) belong to S.

Solution by George Evagelopoulos, Athens, Greece.

Draw a graph whose vertices are the integers $1, 2, \ldots, n$ with an edge between x and y if and only if either (x, y) or (y, x) belongs to S. Those 3-element subsets of S in which any two elements are connected by an edge will be called "triangles" and the number of edges to which a vertex x belongs will be denoted by $d(x)$.



For example, if $S = \{(1,2),(1,3),(2,3),(2,4),(3,4)\}$ where $n = 4$, the subsets $\{(1,2),(1,3),(2,3)\}$ and $\{(2,3),(2,4),(3,4)\}$ form triangles as illustrated. Also $d(2) = 3$, for example. The problem is to determine a lower bound for the number of triangles in the general case.

Let x be joined to y by an edge. Then $d(x) + d(y) \Leftrightarrow 2$ edges are attached to the remaining $n \Leftrightarrow 2$ vertices and therefore at least $d(x)+d(y) \Leftrightarrow 2 \Leftrightarrow (n \Leftrightarrow 2)$ vertices are attached to both x and y. Hence at least $d(x) + d(y) \Leftrightarrow n$ triangles contain both x and y. It follows that the total number of triangles is at least

$$\sum_{(x,y)\in S} \frac{d(x) + d(y) \Leftrightarrow n}{3},$$

since each triangle is counted three times in the sum. Now

$$\sum_{(x,y)\in S} (d(x) + d(y)) = \sum_{x=1}^{n} d(x)^2$$

because each $d(x)$ occurs exactly $d(x)$ times in the sum over the elements of S. Therefore, by Chebyshev's inequality, we get

$$\begin{aligned} \sum_{(x,y)\in S} \frac{d(x) + d(y) \Leftrightarrow n}{3} &= \frac{1}{3}\left(\sum_{x=1}^{n} d(x)^2 \Leftrightarrow nm\right) \\ &\geq \frac{1}{3}\left(\frac{1}{n}\left(\sum_{x=1}^{n} d(x)\right)^2 \Leftrightarrow nm\right) \\ &= \frac{4m(m \Leftrightarrow n^2/4)}{3n} \end{aligned}$$

because $\sum_{x=1}^{n} d(x) = 2m$.

5. Determine all functions f from the reals to the reals for which

(i) $f(x)$ is strictly increasing,

(ii) $f(x)+g(x) = 2x$ for all real x where $g(x)$ is the composition inverse function to $f(x)$. (Note: f and g are said to be composition inverses if $f(g(x)) = x$ and $g(f(x)) = x$ for all real x.)

Solution by George Evagelopoulos, Athens, Greece (with an assist by the editors).

We will prove that $f(x) = x + d$ for some constant d, i.e., $f(x) \Leftrightarrow x$ is a constant function.

For each real d, denote by S_d the set of all x for which $f(x) \Leftrightarrow x = d$. Then we must show that exactly one S_d is nonempty. First we prove two lemmas.

Lemma 1. If $x \in S_d$ then $x + d \in S_d$.

Proof. Suppose $x \in S_d$. Then $f(x) = x+d$, so $g(x+d) = x$, and $f(x+d)+g(x+d) = 2x + 2d$. Therefore $f(x + d) = x + 2d$ and $x + d \in S_d$. $\square$

Lemma 2. If $x \in S_d$ and $y \geq x$ then $y \notin S_{d'}$ for any $d' < d$.

Proof. First let y satisfy $x \leq y < x + (d \Leftrightarrow d')$. Note that by monotonicity $f(y) \geq f(x) = x+d$. Hence $y \in S_{d'}$ would imply $y+d' \geq x+d$ or $y \geq x+(d \Leftrightarrow d')$, a contradiction. Thus $y \notin S_{d'}$ in this case. Now by induction it follows that for all $x \in S_d$,

$$\textit{if } x + (k \Leftrightarrow 1)(d \Leftrightarrow d') \leq y < x + k(d \Leftrightarrow d') \textit{ then } y \notin S_{d'}.$$

The base case $k = 1$ is proved above. Assume the statement holds for some k and let

$$x + k(d \Leftrightarrow d') \leq y < x + (k + 1)(d \Leftrightarrow d').$$

Then

$$x + d + (k \Leftrightarrow 1)(d \Leftrightarrow d') \leq y + d' < x + d + k(d \Leftrightarrow d').$$

But $x + d \in S_d$, and so by the induction hypothesis $y + d' \notin S_{d'}$. The lemma follows. $\square$

Now suppose that two S_d's are nonempty, say S_d and $S_{d'}$ where $d' < d$. If $0 < d'$, then $S_{d'}$ must contain arbitrarily large numbers by Lemma 1. But this is impossible by Lemma 2.

Editor's note. The above, slightly rewritten, is Evagelopoulos's argument, except he hadn't noted that his argument required $0 < d'$. We now complete the proof.

Lemma 3. If S_d and $S_{d'}$ are nonempty and $d' < d'' < d$ then $S_{d''}$ is also nonempty.

Proof. Since S_d and $S_{d'}$ are nonempty, there are x and x' so that $f(x) \Leftrightarrow x = d$ and $f(x') \Leftrightarrow x' = d'$. Since f is increasing and has an inverse, it is continuous, so the function $f(x) \Leftrightarrow x$ is also continuous. Thus by the Intermediate Value Theorem there is x'' so that $f(x'') \Leftrightarrow x'' = d''$, so $S_{d''} \neq \emptyset$. $\square$

Now by Lemma 3 we need only consider two cases: $0 < d' < d$, which was handled by Evagelopoulos, and $d' < d < 0$. We do the second case. Choose some $y \in S_{d'}$. By Lemma 1, S_d contains arbitrarily large *negative* numbers, so we can find $x \in S_d$ such that $x < y$. But then $y \notin S_{d'}$ by Lemma 2. This contradiction completes the proof.

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That's all the room we have this issue. Send me your contests and nice solutions!

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **November 1, 1991,** *although solutions received after that date will also be considered until the time when a solution is published.*

1631*. *Proposed by Murray S. Klamkin, University of Alberta.* (Dedicated to Jack Garfunkel.)

Let P be a point within or on an equilateral triangle and let c_1, c_2, c_3 be the lengths of the three concurrent cevians through P. Determine the largest constant λ such that $c_1^\lambda, c_2^\lambda, c_3^\lambda$ are the sides of a triangle for any P.

1632. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

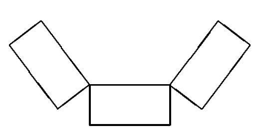
Find all x and y which are rational multiples of π (with $0 < x < y < \pi/2$) such that $\tan x + \tan y = 2$.

1633. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

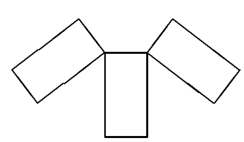
In triangle ABC, the internal bisectors of $\angle B$ and $\angle C$ meet AC and AB at D and E, respectively. We put $\angle BDE = x, \angle CED = y$. Prove that if $\angle A > 60°$ then $\cos 2x + \cos 2y > 1$.

1634. *Proposed by F.F. Nab, Tunnel Mountain, Alberta.*

A cafeteria at a university has round tables (of various sizes) and rectangular trays (all the same size). Diners place their trays of food on the table in one of two ways, depending on whether the short or long sides of the trays point toward the centre of the table:



or



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Moreover, at the same table everybody aligns their trays the same way. Suppose n mathematics students come in to eat together. How should they align their trays so that the table needed is as small as possible?

1635. *Proposed by Jordi Dou, Barcelona, Spain.*

Given points $B_1, C_1, B_2, C_2, B_3, C_3$ in the plane, construct an equilateral triangle $A_1A_2A_3$ so that the triangles $A_1B_1C_1$, $A_2B_2C_2$ and $A_3B_3C_3$ are directly similar.

1636*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Determine the set of all real exponents r such that

$$d_r(x,y) = \frac{|x \Leftrightarrow y|}{(x+y)^r}$$

satisfies the triangle inequality

$$d_r(x,y) + d_r(y,z) \geq d_r(x,z) \quad \text{for all } x,y,z > 0$$

(and thus induces a metric on $\mathbf{R}^+$—see *Crux* 1449, esp. [1990: 224]).

1637. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Prove that

$$\sum \frac{\sin B + \sin C}{A} > \frac{12}{\pi}$$

where the sum is cyclic over the angles A, B, C (measured in radians) of a nonobtuse triangle.

1638. *Proposed by Juan C. Candeal, Universidad de Zaragoza, and Esteban Indurain, Universidad Pública de Navarra, Pamplona, Spain.*

Find all continuous functions $f : (0,\infty) \to (0,\infty)$ satisfying the following two conditions:

(i) f is not one-to-one;

(ii) if $f(x) = f(y)$ then $f(tx) = f(ty)$ for every $t > 0$.

1639. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

$ABCD$ is a convex cyclic quadrilateral. Prove that

$$(AB + CD)^2 + (AD + BC)^2 \geq (AC + BD)^2.$$

1640. *Proposed by P. Penning, Delft, The Netherlands.*

Find

$$\lim_{n\to\infty} \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n \Leftrightarrow 1} \right).$$

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1513. [1990: 43] *Proposed by M.S. Klamkin, University of Alberta.*

(a) A planar centrosymmetric polygon is inscribed in a strictly convex planar centrosymmetric region R. Prove that the two centers coincide.

(b) Do part (a) if the polygon is circumscribed about R.

(c)* Do (a) and (b) still hold if the polygon and region are n-dimensional for $n > 2$?

I. *Solution to* (a) *and* (b) *by Jordi Dou, Barcelona, Spain.*

(a) Let M be the centre of the polygon P and O the centre of R. Let AB be a side of P and ST the symmetrically opposite side with respect to M. Let $A'B'$ be symmetric to AB with respect to O. If O and M are distinct, we have three chords AB, TS and $B'A'$ of R which are of equal length and are parallel, which is incompatible with the strict convexity of R.

(b) Let a be a side of P and s the opposite side. Let a' be symmetric to a with respect to O. If O and M are distinct [and a is chosen not parallel to OM], we have three tangents to R which are mutually parallel, which is again incompatible with the strict convexity of R.

II. *Solution to* (b) *and* (c) *by Marcin E. Kuczma, Warszawa, Poland.*

[Kuczma also solved part (a). — *Ed.*]

(b) Suppose the polygon is circumscribed about R. Choose a pair of parallel sides. They are contained in two parallel supporting lines of R. To every direction there exists exactly one pair of supporting lines, situated symmetrically relative to O, the center of symmetry of R. Thus O lies midway between (the lines containing) the chosen sides. Repeat the argument taking another pair of parallel sides; point O is also equidistant from these sides. The only point with these properties is the center of symmetry of the polygon. Note that strict convexity is not needed in this part.

(c) The analogue of part(b) is true in any dimension $n \geq 2$. The proof is the same as above; just write hyperplanes for lines; and instead of two pairs of opposite sides consider n pairs of opposite faces of the polyhedron.

The analogue of part (a) is **false** in every dimension $n \geq 3$. Here is a counterexample.

Consider the points (in $\mathbf{R}^n$)

$$A_i = (q, \ldots, q, \overset{i\text{th place}}{\overset{\downarrow}{\Leftrightarrow 1}}, q, \ldots, q)\ ,\ \ B_i = (\Leftrightarrow q, \ldots, \Leftrightarrow q, \overset{i\text{th place}}{\overset{\downarrow}{1}}, \Leftrightarrow q, \ldots, \Leftrightarrow q)$$

for $i = 1, \ldots, n$, where q is the root of the equation

$$q^3 + 3q = \frac{4}{n \Leftrightarrow 1}; \tag{1}$$

note that

$$\frac{1}{n \Leftrightarrow 1} < q < 1 \tag{2}$$

since $n \geq 3$.

Let d be the metric in $\mathbf{R}^n$ induced by the 3-norm

$$\|X\|_3 = (\sum_{i=1}^{n} |x_i|^3)^{1/3}$$

for $X = (x_1, \ldots, x_n)$. [*Editor's note*: that is, the "distance" from X to Y is defined to be $\|X \Leftrightarrow Y\|_3$.] By (1),

$$d(A_i, C)^3 = (n \Leftrightarrow 1)(1 \Leftrightarrow q)^3 + 2^3 = (n \Leftrightarrow 1)(1 + 3q^2) + 4 \tag{3}$$

and also

$$d(B_i, C)^3 = (n \Leftrightarrow 1)(1 + q)^3 + 0^3 = (n \Leftrightarrow 1)(1 + 3q^2) + 4. \tag{4}$$

Points $A_1, \ldots, A_n, B_1, \ldots, B_n$ span a polyhedron symmetric about [and so centered at] the origin. Since $A_1, \ldots, A_n$ lie on the hyperplane $\sum X_i = (n \Leftrightarrow 1)q \Leftrightarrow 1$, which by (2) does not pass through the origin, the polyhedron is nondegenerate. In view of (3) and (4) it is inscribed in a certain d-ball, centered at C, *not* at the origin; and this d-ball is certainly a strictly convex centrosymmetric region.

Also solved (part (a)*) by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and (parts* (a) *and* (b)*) the proposer. A further reader sent in an incorrect answer due to misunderstanding the problem.*

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1514. [1990: 44] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle with sides a, b, c and let P be a point in the same plane. Put $AP = R_1, BP = R_2, CP = R_3$. It is well known that there is a triangle with sides aR_1, bR_2, cR_3. Find the locus of P so that the area of this triangle is a given constant.

Solution by Marcin E. Kuczma, Warszawa, Poland.

Inversion with center P and inversion constant k takes points A, B, C to A', B', C' such that

$$\frac{PA'}{PB} = \frac{k}{PA \cdot PB} = \frac{PB'}{PA} \qquad \text{(and cyclically)}.$$

So the triangles PAB and $PB'A'$ are similar, in the above ratio, which is therefore also equal to the ratio $A'B'/AB$. Denoting the sides of triangle $A'B'C'$ by a', b', c' we thus have

$$\frac{a'}{a} = \frac{k}{R_2R_3} \quad , \quad \frac{b'}{b} = \frac{k}{R_3R_1} \quad , \quad \frac{c'}{c} = \frac{k}{R_1R_2}. \tag{1}$$

So if we take

$$k = R_1R_2R_3, \tag{2}$$

the resulting triangle $A'B'C'$ will be the one under consideration.

Let O be the circumcenter of triangle ABC, let R and R' be the circumradii of triangles ABC and $A'B'C'$, and let F and F' be the areas of these triangles. By the properties of inversion (see e.g. N. Altshiller Court, *College Geometry*, New York, 1962, §§526 and 413),

$$\frac{R'}{R} = \frac{k}{|R^2 \Leftrightarrow d^2|}, \tag{3}$$

where $d = OP$ — provided $d \neq R$, of course.

From (1)—(3) and from

$$F = \frac{abc}{4R}\ , \qquad\qquad F' = \frac{a'b'c'}{4R'}$$

we obtain

$$\frac{F'}{F} = |R^2 \Leftrightarrow d^2|\ . \tag{4}$$

Hence F' shall assume a constant value t^2F $(t > 0)$ if and only if $R^2 \Leftrightarrow d^2 = \pm t^2$. So the conclusion is:

if $t > R$, the locus of P is the circle of center O and radius $\sqrt{r^2+t^2}$;

if $t \leq R$, the locus is the union of two circles centered at O and of radii $\sqrt{R^2 \pm t^2}$ (the smaller degenerating to the point O when $t = R$).

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; MURRAY S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

The fact that for some given areas the locus of P consists of two concentric circles was only pointed out by Kuczma and the proposer. Klamkin's proof was rather like the above but obtained equation (4) *by referring to his papers*

Triangle inequalities via transforms, Notices of Amer. Math. Soc.*, January 1972, A–103,104 ; and*

An identity for simplexes and related inequalities, Simon Stevin *48e (1974–75) 57-64;*

or alternatively, to p. 294 of Mitrinović et al, Recent Advances in Geometric Inequalities*, and various places in* Crux*, e.g., [1990: 92].*

*   *   *   *   *

1515. [1990: 44] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

We are given a finite collection of segments in the plane, of total length 1. Prove that there exists a straight line l such that the sum of lengths of projections of the given segments to line l is less than $2/\pi$.

Solution by Jordi Dou, Barcelona, Spain.

We translate the n segments s_i $(1 \leq i \leq n)$ so that their midpoints all coincide at a point V (Figure 1). Designate the $2n$ endpoints, ordered cyclically, by $A_1, A_2, \ldots, A_n$, $A'_1, A'_2, \ldots, A'_n$. Starting at a point P'_n we draw segment P'_nP_1 equal and parallel to VA_1, then starting at P_1 we draw segment P_1P_2 equal and parallel to VA_2, and so on to obtain $P_3, \ldots, P_n, P'_1, \ldots, P'_{n-1}$. We obtain (Figure 2) a convex polygon P of $2n$ sides, with a centre O of symmetry (because the pairs of opposite sides are equal and parallel).

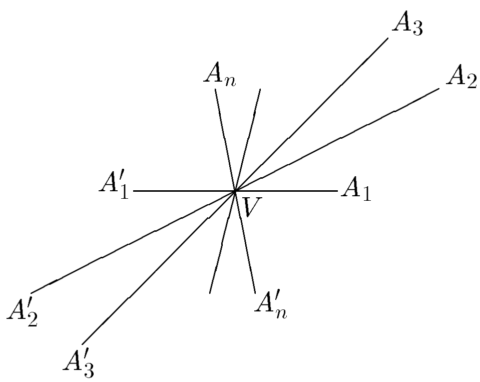


Figure 1

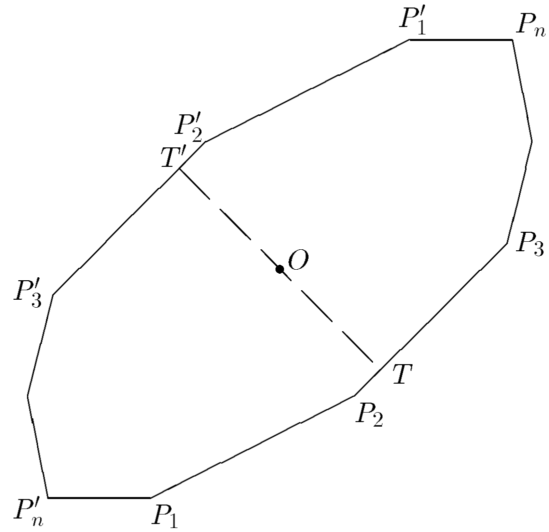


Figure 2

Choose a pair of opposite sides whose distance apart (D) is minimal. Consider the circle ω of centre O and diameter D; it is tangent to the two opposite sides at interior points T and T', and thus is interior to P. Then

$$\pi D < \text{perimeter of } P = \sum_{i=1}^{n} s_i = 1,$$

so $D < 1/\pi$. Therefore the orthogonal projection of all $2n$ sides of P onto TT' has total length $2D < 2/\pi$, as was to be proved.

Note that TT' is the line for which the sum of projections is minimal. It is also clear that the largest diagonal of P gives the direction of maximal sum of projections of the n segments.

Also solved (almost the same way!) by MURRAY S. KLAMKIN, University of Alberta; T. LEINSTER, Lancing College, England; P. PENNING, Delft, The Netherlands; and the proposer.

The proposer did give a second proof using integration, and suggests the analogous problem for segments of total length 1 *in three-dimensional space. With integration he obtains that there is a line so that the sum of the lengths of the projections of the segments to this line is less than* 1/2 *(and this is best possible, as was* $2/\pi$ *in the two-dimensional case). He would like a simpler proof. Might there also be a generalization to planar regions in three-space?*

The problem was suggested by the proposer for the 1989 IMO, but not used (see #73, p. 45 of the book 30th International Mathematical Olympiad, Braunschweig 1989—Problems and Results, *which was reviewed in February).*

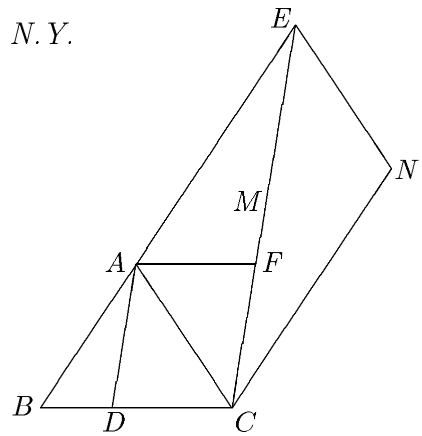
*   *   *   *   *

1516. [1990: 44] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an isosceles triangle in which $AB = AC$ and $\angle A < 90°$. Let D be any point on segment BC. Draw CE parallel to AD meeting AB produced in E. Prove that $CE > 2CD$.

I. *Solution by Jack Garfunkel, Flushing, N.Y.*

Draw AF parallel to BC so that $AF = CD$. Draw $AM = MN$ where M is the midpoint of CE. Then $AENC$ is a parallelogram. Since $\angle A < 90°$, $\angle EAC$ is an obtuse angle. Thus, diagonal $CE >$ diagonal $AN = 2AM$. Also $AM \geq AF$, since a median is $\geq$ the angle bisector drawn from the same vertex. So we have proved the stronger inequality $CE \geq 2AM \geq 2AF = 2CD$.



II. *Solution by T. Leinster, Lancing College, England.*

Triangle CEB is similar to triangle DAB, so $CE = BC \cdot DA/BD$. Thus (since $BC = CD + BD$)

$$\begin{aligned}\frac{CE}{CD} &= \frac{BC \cdot DA}{CD \cdot BD} \geq \frac{BC \cdot DA}{BC^2/4} = \frac{4DA}{BC} \\ &\geq \frac{4}{BC} \cdot \frac{BC}{2} \tan \angle ABC \\ &> 2 \quad (\text{since } \angle ABC > 45°).\end{aligned}$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, and JOSÉ YUSTY PITA, Madrid, Spain; JORDI DOU, Barcelona, Spain; L.J. HUT, Groningen, The Netherlands; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; and the proposer. Two incorrect solutions were also sent in.

Kuczma's solution was very similar to solution II. *Several solvers pointed out that $CE = 2CD$ holds if $\angle A = 90°$ and D is the midpoint of BC.*

* * * * *

1517*. [1990: 44] *Proposed by Bill Sands, University of Calgary.*

[This problem came up in a combinatorics course, and is quite likely already known. What is wanted is a nice answer with a nice proof. A reference would also be welcome.]

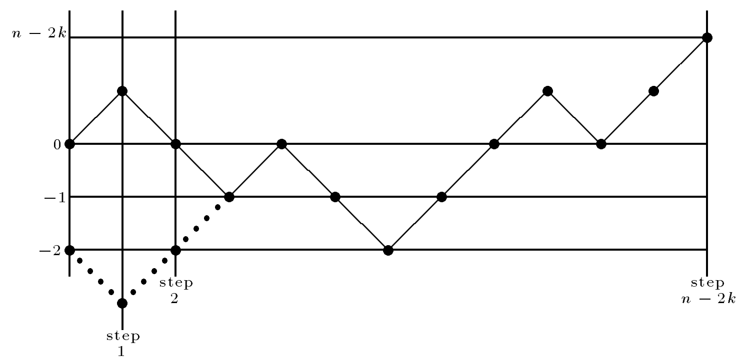
Imagine you are standing at a point on the edge of a half-plane extending infinitely far north, east, and west (say, on the Canada–USA border near Estevan, Saskatchewan). How many walks of n steps can you make, if each step is 1 metre either north, east, west, or south, and you never step off the half-plane? For example, there are three such walks of length 1 and ten of length 2.

I. *Solution by Mike Hirschhorn, University of New South Wales, Kensington, Australia.*

First we note: the number of walks of n (unit) steps on $\mathbf{Z}^+$ (the nonnegative integers) starting at 0 and ending at $n-2k$ is

$$\binom{n}{k} - \binom{n}{k-1}, \qquad 0 \le k \le [n/2]. \tag{1}$$

[*Editor's note.* This can be proved by the familiar (but worth repeating) "reflection principle". The total number of n-step walks on $\mathbf{Z}$ starting at 0 and ending at $n-2k$ equals the number of n-sequences of $n-k$ R's (rights) and k L's (lefts), which is $\binom{n}{k}$. From this we wish to subtract the "bad" walks, i.e., the n-step walks from 0 to $n-2k$ which land on -1 at some point. By the reflection principle these correspond to the n-walks from -2 to $n-2k$, i.e., the number of n-sequences of $n-k+1$ R's and $k-1$ L's; there are $\binom{n}{k-1}$ of these, so (1) follows. The correspondence of the above walks is obtained by interchanging R's and L's on that part of the walk from 0 to the first time -1 is reached. Drawing walks as "graphs" (as shown), this amounts to "reflecting" an initial piece of the graph about the horizontal line $y = -1$, giving the dotted graph. Back to Hirschhorn's proof!]



A corollary is: the number of walks of n steps on $\mathbf{Z}^+$ starting at 0 is

$$\sum_{k=0}^{[n/2]} \left[\binom{n}{k} - \binom{n}{k-1} \right] = \binom{n}{[n/2]} \tag{2}$$

(follows by telescoping).

Finally we prove that walks of n unit steps on $\mathbf{Z} \times \mathbf{Z}^+$ (i.e., the integer points of the upper half-plane) starting at $(0,0)$ are equinumerous with walks of $2n+1$ steps on $\mathbf{Z}^+$ starting at 0.

To see this, consider a walk of n steps on $\mathbf{Z} \times \mathbf{Z}^+$. It can be written as a sequence of moves L, R, U(up), D(down). [At each stage the number of U's must equal or exceed the number of D's.] Replace each L by LR, R by RL, U by RR, D by LL, and prefix this sequence with an R. This process gives a walk of length $2n+1$ on $\mathbf{Z}^+$, and the process is reversible.

Therefore the number of walks of n steps on $\mathbf{Z} \times \mathbf{Z}^+$ starting at $(0,0)$ is, by (2),

$$\binom{2n+1}{n}.$$

[*Editor's note.* Hirschhorn sent in a second proof, based on a formula for the number of walks of n steps on $\mathbf{Z} \times \mathbf{Z}^+$ from $(0,0)$ to (x,y).]

II. *Combination of solutions by H.L. Abbott, University of Alberta, and by Chris Wildhagen, Breda, The Netherlands.*

Let the half-plane be given by $\{(x,y) : y \geq 0\}$ and let $g_k(n)$ denote the number of walks with n steps which have initial point $(0,k)$ and which do not leave the half-plane. The problem calls for $g_0(n)$. Observe that for $k \geq 1$,

$$g_k(n) = 2g_k(n \Leftrightarrow 1) + g_{k+1}(n \Leftrightarrow 1) + g_{k-1}(n \Leftrightarrow 1). \tag{3}$$

In fact, (3) follows from the observation that the walks counted by $g_k(n)$ may be split into four sets according to the direction of the first step. Similarly, one gets

$$g_0(n) = 2g_0(n \Leftrightarrow 1) + g_1(n \Leftrightarrow 1). \tag{4}$$

We have the boundary conditions

$$g_0(1) = 3 \quad \text{and} \quad g_k(n) = 4^n \text{ for } k \geq n. \tag{5}$$

It is clear that $g_k(n)$ is uniquely determined by (3), (4) and (5). We shall prove by induction on n that

$$g_k(n) = \sum_{j=0}^{k} \binom{2n+1}{n \Leftrightarrow j}$$

satisfies (3), (4) and (5). $g_0(1) = 3$ is obvious, and for $k \geq n$

$$\sum_{j=1}^{k} \binom{2n+1}{n \Leftrightarrow j} = \sum_{i=0}^{n} \binom{2n+1}{i} = \frac{1}{2} \sum_{i=0}^{2n+1} \binom{2n+1}{i} = \frac{1}{2} 2^{2n+1} = 4^n,$$

so (5) holds. For (4) we have

$$\begin{aligned} g_0(n) &= 2\binom{2n \Leftrightarrow 1}{n \Leftrightarrow 1} + \binom{2n \Leftrightarrow 1}{n \Leftrightarrow 1} + \binom{2n \Leftrightarrow 1}{n \Leftrightarrow 2} \\ &= \binom{2n \Leftrightarrow 1}{n} + 2\binom{2n \Leftrightarrow 1}{n \Leftrightarrow 1} + \binom{2n \Leftrightarrow 1}{n \Leftrightarrow 2} \\ &= \binom{2n}{n} + \binom{2n}{n \Leftrightarrow 1} = \binom{2n+1}{n}, \end{aligned}$$

and for (3)

$$\begin{aligned} g_k(n) &= 2\sum_{j=0}^{k} \binom{2n \Leftrightarrow 1}{n \Leftrightarrow 1 \Leftrightarrow j} + \sum_{j=0}^{k+1} \binom{2n \Leftrightarrow 1}{n \Leftrightarrow 1 \Leftrightarrow j} + \sum_{j=0}^{k-1} \binom{2n \Leftrightarrow 1}{n \Leftrightarrow 1 \Leftrightarrow j} \\ &= 2\sum_{j=0}^{k} \binom{2n \Leftrightarrow 1}{n \Leftrightarrow 1 \Leftrightarrow j} + \sum_{j=0}^{k} \binom{2n \Leftrightarrow 1}{n \Leftrightarrow 2 \Leftrightarrow j} + \binom{2n \Leftrightarrow 1}{n \Leftrightarrow 1} + \sum_{j=0}^{k} \binom{2n \Leftrightarrow 1}{n \Leftrightarrow j} \Leftrightarrow \binom{2n \Leftrightarrow 1}{n} \\ &= \sum_{j=0}^{k} \left[\binom{2n \Leftrightarrow 1}{n \Leftrightarrow 1 \Leftrightarrow j} + \binom{2n \Leftrightarrow 1}{n \Leftrightarrow 2 \Leftrightarrow j} \right] + \sum_{j=0}^{k} \left[\binom{2n \Leftrightarrow 1}{n \Leftrightarrow 1 \Leftrightarrow j} + \binom{2n \Leftrightarrow 1}{n \Leftrightarrow j} \right] \\ &= \sum_{j=0}^{k} \binom{2n}{n \Leftrightarrow 1 \Leftrightarrow j} + \sum_{j=0}^{k} \binom{2n}{n \Leftrightarrow j} = \sum_{j=0}^{k} \binom{2n+1}{n \Leftrightarrow j}. \end{aligned}$$

In particular, the required number of walks is

$$g_0(n) = \binom{2n+1}{n}.$$

Also solved by W. BRECKENRIDGE (student), H. GASTINEAU-HILLS, A. NELSON, P. BOS (student), G. CALVERT (student) and K. WEHRHAHN, all of the University of Sydney, Australia; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD K. GUY, University of Calgary; GEORGE P. HENDERSON, Campbellcroft, Ontario; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; CHRISTIAN KRATTENTHALER, Institut für Mathematik, Vienna, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; BRUCE E. SAGAN, Michigan State University, East Lansing; ROBERT E. SHAFER, Berkeley, California; and GEORGE SZEKERES, University of New South Wales, Kensington, Australia. Two other readers sent in the correct answer, one with no proof and one with an incorrect proof.

Breckenridge et al have written an article, "Lattice paths and Catalan numbers", which has appeared in the Bulletin of the Institute of Combinatorics and its Applications, *Vol. 1 (1991), pp. 41–55, dealing with this problem and related questions. In this paper they give a proof of the present problem which is very similar to solution* I.

Guy, Krattenthaler and Sagan have also written a paper, "Lattice paths, reflections, and dimension-changing bijections" (submitted for publication), in which they solve this problem and tackle several analogous problems in the plane and in higher dimensions. Guy has compiled separately an extensive annotated list of references pertaining to related topics.

To the editor's surprise, **no** *reference to the precise problem asked could be supplied by the readers.*

*   *   *   *   *

1518. [1990: 44] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

O and H are respectively the circumcenter and orthocenter of a triangle ABC in which $\angle A \neq 90°$. Characterize triangles ABC for which ΔAOH is isosceles. Which of these triangles ABC have integer sides?

Combined solutions of Marcin E. Kuczma, Warszawa, Poland; and the proposer.

Triangle AOH can be isosceles in three ways. In each case, the reasoning is written in the form of successively equivalent statements; the first line identifies the case and the last line gives the desired characterization of triangle ABC. The vector equality

$$\vec{OH} = \vec{OA} + \vec{OB} + \vec{OC}$$

is the clue. R denotes the circumradius, and sums are cyclic over A, B, C.

Case (i): $AH = AO$

$$\Leftrightarrow \quad |\,\vec{OH} \Leftrightarrow \vec{OA}\,| = |\,\vec{OA}\,| \quad \Leftrightarrow \quad (\vec{OB} + \vec{OC})^2 = (\vec{OA})^2$$

$$\Leftrightarrow \quad 2R^2 + 2\,\vec{OB} \cdot \vec{OC} = R^2 \quad \Leftrightarrow \quad 2\cos 2A = \Leftrightarrow 1$$

$$\Leftrightarrow \quad \angle A = 60° \text{ or } 120°.$$

Case (ii): $OH = OA$

$$\begin{aligned}
&\Leftrightarrow \quad (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})^2 = (\overrightarrow{OA})^2 \quad \Leftrightarrow \quad 3R^2 + 2\sum \overrightarrow{OB} \cdot \overrightarrow{OC} = R^2 \\
&\Leftrightarrow \quad \sum \cos 2a = \Leftrightarrow 1 \quad \Leftrightarrow \quad \cos A \cos B \cos C = 0 \\
&\qquad\qquad [\text{using } 4\cos A \cos B \cos C = 1 + \sum \cos 2A] \\
&\Leftrightarrow \quad \angle B = 90^\circ \text{ or } \angle C = 90^\circ.
\end{aligned}$$

Case (iii): $HO = HA$

$$\begin{aligned}
&\Leftrightarrow \quad |\,\overrightarrow{OH}\,| = |\,\overrightarrow{OH} \Leftrightarrow \overrightarrow{OA}\,| \quad \Leftrightarrow \quad (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})^2 = (\overrightarrow{OB} + \overrightarrow{OC})^2 \\
&\Leftrightarrow \quad 3R^2 + 2\sum \overrightarrow{OB} \cdot \overrightarrow{OC} = 2R^2 + 2\,\overrightarrow{OB} \cdot \overrightarrow{OC} \\
&\Leftrightarrow \quad 2(\cos 2B + \cos 2C) = \Leftrightarrow 1 \\
&\Leftrightarrow \quad 4\cos A \cos(B \Leftrightarrow C) = 1
\end{aligned}$$

[using $\cos X + \cos Y = 2\cos\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right)$ with $X = 2B$, $Y = 2C$]. This solves the first part of the problem.

To cope with the last question, we must restate the resultant equalities in terms of side lengths. In case (i) we are led to the alternatives

$$a^2 = b^2 \Leftrightarrow bc + c^2 \qquad \text{or} \qquad a^2 = b^2 + bc + c^2; \tag{1}$$

in case (ii) to

$$b^2 = a^2 + c^2 \qquad \text{or} \qquad c^2 = a^2 + b^2. \tag{2}$$

If $a^2 = b^2 \Leftrightarrow bc + c^2$ in (1), without loss of generality let $b > c$. Then $a^2 \Leftrightarrow c^2 = b^2 \Leftrightarrow bc$ so that

$$\frac{a \Leftrightarrow c}{b \Leftrightarrow c} = \frac{b}{a+c} = \frac{m}{n}$$

in lowest terms. Thus

$$a \Leftrightarrow c = \frac{m}{n}(b \Leftrightarrow c) \qquad \text{or} \qquad a + \left(\frac{m \Leftrightarrow n}{n}\right) c = \left(\frac{m}{n}\right) b$$

and

$$a + c = \left(\frac{n}{m}\right) b.$$

Solving for a and c, we have

$$a = \left(\frac{m^2 \Leftrightarrow mn + n^2}{2mn \Leftrightarrow m^2}\right) b\ , \qquad c = \left(\frac{n^2 \Leftrightarrow m^2}{2mn \Leftrightarrow m^2}\right) b\ .$$

Thus

$$a = k(m^2 \Leftrightarrow mn + n^2)\ ,\ b = k(2mn \Leftrightarrow m^2)\ ,\ c = k(n^2 \Leftrightarrow m^2). \tag{3}$$

Similarly from $a^2 = b^2 + bc + c^2$ we obtain

$$a = k(m^2 + mn + n^2)\ ,\ b = k(2mn + m^2)\ ,\ c = k(n^2 \Leftrightarrow m^2). \tag{4}$$

As m, n range over all pairs of relatively prime integers with $n > m > 0$, the sets of formulas in (3) and (4) for $k = 1$ produce triples of positive integers a, b, c which are either mutually coprime or divisible by 3; after possible reduction by 3 we obtain the complete solution to the equations in (1) in irreducible triples of positive integers.

For (2) the irreducible solutions are the Pythagorean triples

$$a = 2xy\ ,\ \ b = x^2 + y^2\ ,\ \ c = x^2 - y^2$$

or

$$a = 2xy\ ,\ \ b = x^2 - y^2\ ,\ \ c = x^2 + y^2,$$

with a possible reduction factor of 2.

For case (iii) we have

$$\begin{aligned} 4\cos A\cos(B - C) = 1 \quad &\Leftrightarrow \quad 4\cos(B + C)\cos(B - C) = -1 \\ &\Leftrightarrow \quad 2(\cos 2B + \cos 2C) = -1 \\ &\Leftrightarrow \quad 4\cos^2 B + 4\cos^2 C = 3. \end{aligned} \tag{5}$$

By the cosine formula, $2\cos B$ and $2\cos C$ are rational in a triangle with integer sides, so (5) can be written in the form

$$x^2 + y^2 = 3z^2$$

for x, y, z integral and not all even, which, by reducing modulo 4, clearly has no solution. Thus there are no solutions in case (iii).

There were two partial solutions submitted.

The proposer's original problem actually contained the additional condition that the area of the triangle also be integral; as the proposer points out, case (i) *above then has no solution, since* $\sin 60°$ *is irrational, and so the only solutions are the right-angled triangles of case* (ii). *The editor, however, inadvertently omitted the area condition from the problem.*

* * * * *

1519. [1990: 44] *Proposed by P. Penning, Delft, The Netherlands.*

Find all prime numbers which, written in the number system with base b, contain each digit $0, 1, \ldots, b-1$ exactly once (a leading zero is allowed).

Solution by Richard I. Hess, Rancho Palos Verdes, California.

If p is such a prime number, with base b representation $(a_{b-1}\ a_{b-2}\cdots a_1 a_0)_b$, then

$$\begin{aligned} p &= a_0 + a_1 b + a_2 b^2 + \cdots + a_{b-1} b^{b-1} \\ &= (a_0 + a_1 + \cdots + a_{b-1}) + a_1(b - 1) + a_2(b^2 - 1) + \cdots + a_{b-1}(b^{b-1} - 1) \\ &= \frac{b(b-1)}{2} + a_1(b - 1) + a_2(b^2 - 1) + \cdots + a_{b-1}(b^{b-1} - 1). \end{aligned}$$

Thus for $b = 2k$ p will be divisible by $b \Leftrightarrow 1$, so the only such primes for even bases are for $b = 2$; in fact, only $2 = (10)_2$. For $b = 2k+1$ p will be divisible by k, so the only such primes for odd bases are for $b = 3$, and we find the solutions $(012)_3 = 5, (021)_3 = 7, (102)_3 = 11$, and $(201)_3 = 19$.

Also solved by H.L. ABBOTT, University of Alberta; DUANE M. BROLINE, Eastern Illinois University, Charleston; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MARCIN E. KUCZMA, Warszawa, Poland; T. LEINSTER, Lancing College, England; R.E. SHAFER, Berkeley, California; ST. OLAF PROBLEM SOLVING GROUP, St. Olaf College, Northfield, Minnesota; CHRIS WILDHAGEN, Breda, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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1520. [1990: 44] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

A point is said to be *inside* a parabola if it is on the same side of the parabola as the focus. Given a finite number of parabolas in the plane, must there be some point of the plane that is not inside any of the parabolas?

Solution by Jordi Dou, Barcelona, Spain.

Let ℓ be a line not parallel to any of the axes of the given parabolas $P_1, \ldots, P_n$.

The interior points of P_i that belong to ℓ , if there are any, form a segment $I_i = A_iB_i$ where $\ell \cap P_i = \{A_i, B_i\}$. The set of points of ℓ interior to any of the n parabolas is then the union $\cup_{j=1}^n I_i$, which consists of at most n segments of finite total length, while the remainder, points exterior to all the parabolas P_i, have infinite length.

Another proof is as follows. Given n parabolas and a point O, let Ω be a circle of centre O and variable radius x. Denote by $I(x)$ the area of the region, contained in $\Omega(x)$, formed by points interior to any parabola, and by $E(x)$ the area of the complementary region $\Omega(x) \Leftrightarrow I(x)$, formed by points exterior to all the parabolas. So $I(x) + E(x) = \pi x^2$. Then we can prove that $\lim_{x\to\infty}(I(x)/E(x)) = 0$. (The reason is that $I(x)$ is of order $x\sqrt{x}$, while $E(x)$ is of order x^2.)

Also proved (usually by one or the other of the above methods) by H.L. ABBOTT, University of Alberta; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; T. LEINSTER, Lancing College, England; R.C. LYNESS, Southwold, England; P. PENNING, Delft, The Netherlands; DAVID SINGMASTER, South Bank Polytechnic, London, England; ST. OLAF PROBLEM SOLVING GROUP, St. Olaf College, Northfield, Minnesota; and the proposer. A comment was also received from a reader who evidently didn't understand the problem.

One of the above solvers (whose identity will be kept secret) claims that the same result is true for a countable *number of parabolas. Is it?*

This is the third *solution of Jordi Dou to be featured in this issue. It seems fitting to disclose here that Dou recently wrote to the editor, and (while conveying 80th birthday greetings to Dan Pedoe) mentioned that he had himself just turned 80 on June 7!* Crux *readers, and certainly this admiring and appreciative editor, would wish Professor Dou all the best.*

* * * * *

1521. [1990:74] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Triangle ABC has angles α, β, γ, circumcenter O, incenter I, and orthocenter H. Suppose that the points A, H, I, O, B are concyclic.

(a) Find γ.

(b) Prove $HI = IO$.

(c) If $AH = HI$, find α and β.

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

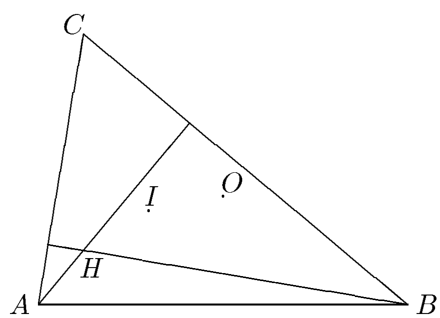
(a) If A, H, I, O, B are concyclic, then

$$\angle AHB = \angle AIB = \angle AOB,$$

so

$$180^\circ \Leftrightarrow \gamma = 90^\circ + \frac{\gamma}{2} = 2\gamma,$$

so $\gamma = 60^\circ$.



Remark: if two of the three points H, I, O are concyclic with A and B, then the third one is as well. [*Editor's note*. This assumes ΔABC is acute. If $\angle C = 120^\circ$ then A, H, O, B are concyclic but I does not lie on the same circle.]

(b) From $\angle ABH = 90^\circ \Leftrightarrow \alpha = \angle OBC$ we obtain

$$\angle OBI = \frac{1}{2}(\alpha \Leftrightarrow \gamma) = \angle IBH,$$

so $HI = IO$.

(c) If $AH = HI$ then

$$90^\circ \Leftrightarrow \alpha = \angle ABH = \angle HBI = \frac{\alpha}{2} \Leftrightarrow \frac{\gamma}{2} = \frac{\alpha}{2} \Leftrightarrow 30^\circ,$$

so $\alpha = 80^\circ$ and thus $\beta = 40^\circ$.

Also solved by WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, and JOSÉ YUSTY PITA, Madrid, Spain; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; HIDETOSI FUKAGAWA, Aichi, Japan; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong;

TOM LEINSTER, Lancing College, England; P. PENNING, Delft, The Netherlands; K.R.S. SASTRY, Addis Ababa, Ethiopia; TOSHIO SEIMIYA, Kawasaki, Japan; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

Engelhaupt, Fukagawa and Kuczma observed that it need only be assumed that A, I, O, B are concyclic, Leinster the same when A, H, I, B are concyclic. Dou and Sastry give converses. Dou also notes that under the hypothesis of the problem, segments OH and CO' are perpendicular bisectors of each other, where O' is the centre of circle $ABOIH$. The proposer also proved that, letting S be the second intersection of BC with circle $ABOIH$, IS is parallel to AB and $IS = AI = SB$.

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1522. [1990: 74] *Proposed by M.S. Klamkin, University of Alberta.*

Show that if $a, b, c, d, x, y > 0$ and

$$xy = ac + bd \quad , \quad \frac{x}{y} = \frac{ad+bc}{ab+cd} \ ,$$

then

$$\frac{abx}{a+b+x} + \frac{cdx}{c+d+x} = \frac{ady}{a+d+y} + \frac{bcy}{b+c+y}.$$

I. *Solution by Seung-Jin Bang, Seoul, Republic of Korea.*

Note that, from the given expressions for xy and x/y,

$$ab(c+d+x) + cd(a+b+x) = ad(b+c+y) + bc(a+d+y)$$

and

$$\begin{aligned} x(a+d+y)(b+c+y) &= ((a+d)x + ac + bd)(b+c+y) \\ &= (ac+bd)y + (a+d)(b+c)x + (b+c)(ac+bd) + (a+d)(ac+bd) \\ &= (ac+bd)x + (a+b)(c+d)y + (c+d)(ac+bd) + (a+b)(ac+bd) \\ &= ((a+b)y + ac + bd)(c+d+x) \\ &= y(a+b+x)(c+d+x). \end{aligned}$$

It follows that

$$x\left(\frac{ab}{a+b+x} + \frac{cd}{c+d+x}\right) = y\left(\frac{ad}{a+d+y} + \frac{bc}{b+c+y}\right).$$

II. *Combination of partial solutions by Wilson da Costa Areias, Rio de Janeiro, Brazil, and Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

It is well known (see p. 111, no. 207 of N. Altshiller-Court, *College Geometry*) that it is possible to construct a cyclic quadrilateral $ABCD$ of sides $AB = a$, $BC = b$, $CD = c$, $DA = d$ [provided that $a < b + c + d$, etc. — *Ed.*] and also that, from Ptolemy's theorem, its diagonals $x = AC$, $y = BD$ satisfy

$$xy = ac + bd \ , \qquad \frac{x}{y} = \frac{ad + bc}{ab + cd} \ .$$

Let R be the circumradius of $ABCD$, and let F_1, F_2, F_3, F_4 and s_1, s_2, s_3, s_4 be the areas and semiperimeters of the triangles ABC, BCD, CDA, DAB, respectively. Then

$$\frac{abx}{a+b+x} + \frac{cdx}{c+d+x} = \frac{bcy}{b+c+y} + \frac{ady}{a+d+y}$$

is equivalent to

$$\frac{F_1 \cdot 4R}{2s_1} + \frac{F_3 \cdot 4R}{2s_3} = \frac{F_2 \cdot 4R}{2s_2} + \frac{F_4 \cdot 4R}{2s_4},$$

or

$$r_1 + r_3 = \frac{F_1}{s_1} + \frac{F_3}{s_3} = \frac{F_2}{s_2} + \frac{F_4}{s_4} = r_2 + r_4,$$

where r_1, r_2, r_3, r_4 are respectively the inradii of the above triangles. The relation $r_1 + r_3 = r_2 + r_4$ is true and has been shown by H. Forder in "An ancient Chinese theorem", Math Note 2128, p. 68 of *Mathematical Gazette* 34, no. 307 (1950). It was also part (b) of *Crux* 1226 [1988: 147].

Also solved by DUANE M. BROLINE, Eastern Illinois University, Charleston; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; GEORGE P. HENDERSON, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; TOM LEINSTER, Lancing College, England; J.A. MCCALLUM, Medicine Hat, Alberta; VEDULA N. MURTY, Penn State University at Harrisburg; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Murty points out that the result holds for all real a, b, c, d, x, y provided only that the denominators in the problem are nonzero (as is clear from solution I*).*

The geometrical interpretation given in (partial) solution II *was also noted by Festraels-Hamoir, Kuczma, Smeenk and the proposer. Most of them, however, gave algebraic proofs, thus avoiding the cases (e.g., $a > b + c + d$) missed by Areias and Bellot. Kuczma gave a separate and rather involved argument to cover these cases. Is there a "nice" way to complete solution* II*?*

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