Crux Mathematicorum

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CONTENTS

THE OLYMPIAD CORNER

No. 125

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We begin this month with the problems of the 40th Mathematical Olympiad from Poland. This was written in April 1989. My thanks to Marcin E. Kuczma for sending the problem set to me.

40th MATHEMATICAL OLYMPIAD IN POLAND

Final round (April, 1989)

1. An even number of people are participating in a round table conference. After lunch break the participants change seats. Show that some two persons are separated by the same number of persons as they were before break.

2. K_1, K_2, K_3 are pairwise externally tangent circles in the plane. K_2 touches K_3 at P, K_3 touches K_1 at Q, K_1 touches K_2 at R. Lines PQ and PR cut K_1 in the respective points S and T (other than Q and R). Lines SR and TQ cut K_2 and K_3 in the respective points U and V (other than R and Q). Prove that P lies in line with U and V.

3. The edges of a cube are numbered 1 through 12.

(a) Prove that for every such numbering there exist at least eight triples of integers (i, j, k) with $1 \le i < j < k \le 12$ such that the edges assigned numbers i, j, k are consecutive segments of a polygonal line.

(b) Give an example of a numbering for which a ninth triple with properties stated in (a) does not exist.

4. Let n and k be given positive integers. Consider a chain of sets $A_0, \ldots, A_k$ in which $A_0 = \{1, \ldots, n\}$ and, for each i ($i = 1, \ldots, k$), A_i is a randomly chosen subset of A_{i-1}; all choices are equiprobable. Show that the expected cardinality of A_k is $n2^{-k}$.

5. Three pairwise tangent circles of equal radius a lie on a hemisphere of radius r. Determine the radius of a fourth circle contained in the same sphere and tangent to the three given ones.

6. Prove that the inequality

$$\left(\frac{ab + ac + ad + bc + bd + cd}{6}\right)^{1/2} \ge \left(\frac{abc + abd + acd + bcd}{4}\right)^{1/3}$$

holds for any positive numbers a, b, c, d.

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A second Olympiad for this issue comes from Sweden via Andy Liu.

SWEDISH MATHEMATICAL COMPETITION

Final round: November 18, 1989
Time: 5 hours

1. Let n be a positive integer. Prove that the integers $n^2(n^2+2)^2$ and $n^4(n^2+2)^2$ can be written in base n^2+1 with the same digits but in opposite order.

2. Determine all continuous functions f such that $f(x)+f(x^2)=0$ for all real numbers x.

3. Find all positive integers n such that $n^3-18n^2+115n-391$ is the cube of a positive integer.

4. Let $ABCD$ be a regular tetrahedron. Where on the edge BD is the point P situated if the edge CD is tangent to the sphere with diameter AP?

5. Assume that $x_1,\ldots,x_5$ are positive real numbers such that $x_1<x_2$ and assume that x_3,x_4,x_5 are all greater than x_2. Prove that if $\alpha>0$, then

$$\frac{1}{(x_1+x_3)^\alpha}+\frac{1}{(x_2+x_4)^\alpha}+\frac{1}{(x_2+x_5)^\alpha}<\frac{1}{(x_1+x_2)^\alpha}+\frac{1}{(x_2+x_3)^\alpha}+\frac{1}{(x_4+x_5)^\alpha}\ .$$

6. On a circle $4n$ points, $n\geq 1$, are chosen. Every second point is colored yellow, the other points are colored blue. The yellow points are divided into n pairs and the points in each pair are connected with a yellow line segment. In the same manner the blue points are divided into n pairs and the points in each pair are connected with a blue line segment. Assume that at most two line segments pass through each point in the interior of the circle. Prove that there are at least n points of intersection of blue and yellow line segments.

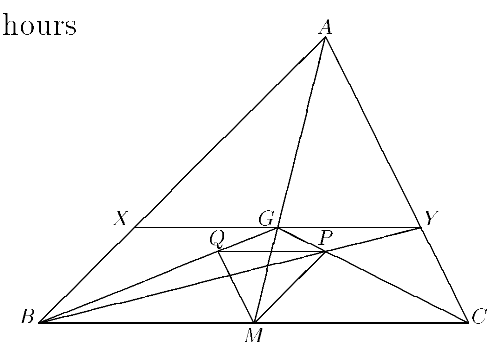
*

It has become a custom to give the problems of the Asian Pacific Mathematics Olympiad. Since "official solutions" for this contest are widely distributed we will only publish particularly novel and interesting solutions. My thanks to Ed Barbeau and Andy Liu for sending me this problem set.

1991 ASIAN PACIFIC MATHEMATICS OLYMPIAD

March 1991
Time allowed: 4 hours

1. Given ΔABC, let G be the centroid and M be the mid-point of BC. Let X be on AB and Y on AC such that the points X, G and Y are collinear and XGY and BC are parallel. Suppose that XC and GB intersect at Q and that YB and GC intersect at P. Show that ΔMPQ is similar to ΔABC.



2. Suppose there are 997 points given on a plane. If every two points are joined by a line segment with its mid-point coloured in red, show that there are at least 1991 red points on the plane. Can you find a special case with exactly 1991 red points?

3. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers such that

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k \ .$$

Show that

$$\sum_{k=1}^{n} \frac{(a_k)^2}{a_k + b_k} \geq \frac{1}{2} \ \sum_{k=1}^{n} a_k \ .$$

4. During a break n children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule: he selects one child and gives him a candy, then he skips the next child and gives a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and so on. Determine the values of n for which eventually (perhaps after many rounds) all children will have at least one candy each.

5. Given are two tangent circles, C_1, C_2, and a point P on their radical axis, i.e. on the common tangent of C_1 and C_2 that is perpendicular to the line joining the centres of C_1 and C_2. Construct with compass and ruler all the circles C that are tangent to C_1 and C_2 and pass through the point P.

*

I am giving more problems than usual this issue to give readers some sources of pleasure for the summer break. Also the June issue is normally taken up by two contests for which we do not usually publish readers' solutions.

* * *

Last issue we gave the problems of the A.I.M.E. for 1991. As promised, we next give the numerical solutions. The problems and their official solutions are copyrighted by the Committee of the American Mathematics Competitions of the Mathematical Association of America, and may not be reproduced without permission. Detailed solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, Nebraska, U.S.A., 68588-0322.

1. 146	2. 840	3. 166	4. 159	5. 128
6. 743	7. 383	8. 010	9. 044	10. 532
11. 135	12. 677	13. 990	14. 384	15. 012

* * *

We now turn to some further solutions submitted by readers for problems from the 'Archives'. First a problem from April 1984.

3. [1984: 108] *1982 Austrian-Polish Mathematics Competition.*
Prove that, for all natural numbers $n \geq 2$,

$$\prod_{i=1}^{n} \tan\left\{\frac{\pi}{3}\left(1+\frac{3^i}{3^n-1}\right)\right\} = \prod_{i=1}^{n} \cot\left\{\frac{\pi}{3}\left(1-\frac{3^i}{3^n-1}\right)\right\} .$$

Solution by Murray S. Klamkin, University of Alberta.
Let

$$A_i = \tan\left\{\frac{\pi}{3}\left(1+\frac{3^i}{3^n-1}\right)\right\} = \frac{\tan\frac{\pi}{3} + \tan(\frac{\pi 3^{i-1}}{3^n-1})}{1-\tan\frac{\pi}{3}\tan(\frac{\pi 3^{i-1}}{3^n-1})}$$

and

$$B_i = \tan\left\{\frac{\pi}{3}\left(1-\frac{3^i}{3^n-1}\right)\right\} = \frac{\tan\frac{\pi}{3} - \tan(\frac{\pi 3^{i-1}}{3^n-1})}{1+\tan\frac{\pi}{3}\tan(\frac{\pi 3^{i-1}}{3^n-1})}.$$

Now since

$$\tan 3\theta = \tan\theta\left[\frac{3-\tan^2\theta}{1-3\tan^2\theta}\right]$$

we get

$$A_iB_i = \frac{3-\tan^2(\frac{\pi 3^{i-1}}{3^n-1})}{1-3\tan^2(\frac{\pi 3^{i-1}}{3^n-1})} = \frac{\tan(\frac{\pi 3^{i}}{3^n-1})}{\tan(\frac{\pi 3^{i-1}}{3^n-1})}.$$

Hence

$$\prod_{i=1}^{n} A_iB_i = \frac{\tan(\frac{\pi 3^{n}}{3^n-1})}{\tan(\frac{\pi}{3^n-1})} = \frac{\tan(\pi+\frac{\pi}{3^n-1})}{\tan\left(\frac{\pi}{3^n-1}\right)} = 1.$$

From this the result is immediate.

*

4. [1986: 19] *1985 Austria-Poland Mathematical Competition.*
Determine all real solutions x, y of the system

$$x^4 + y^2 - xy^3 - 9x/8 = 0,$$
$$y^4 + x^2 - yx^3 - 9y/8 = 0.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.
We find instead all the solutions of the given equations, which we rewrite as

$$8x^4 + 8y^2 - 8xy^3 - 9x = 0 , \qquad (1)$$
$$8y^4 + 8x^2 - 8yx^3 - 9y = 0 . \qquad (2)$$

We show there are exactly *ten* real or complex solutions, namely

$$(0,0), (\frac{9}{8}, \frac{9}{8}), (1, \frac{1}{2}), (\frac{1}{2}, 1), (\omega, \frac{\omega^2}{2}), (\omega^2, \frac{\omega}{2})(\frac{\omega}{2}, \omega^2), (\frac{\omega^2}{2}, \omega), (\frac{9\omega}{8}, \frac{9\omega^2}{8}), (\frac{9\omega^2}{8}, \frac{9\omega}{8}),$$

where $\omega = (-1+\sqrt{3}i)/2$ denotes a complex cube root of unity.

From (y times (1)) minus (x times (2)) we get

$$\begin{aligned} 0 &= 2(x^4y - xy^4) - (x^3 - y^3) = (x^3 - y^3)(2xy - 1) \\ &= (x-y)(2xy-1)(x^2+xy+y^2) \ . \end{aligned}$$

If $x = y$, then substitution in (1) yields $8x^2 - 9x = 0$ which immediately gives two candidates $(0,0)$ and $(9/8, 9/8)$. If $2xy = 1$, then substituting $y = (2x)^{-1}$ in (1) and simplifying we obtain

$$8x^4 + \frac{1}{x^2} - 9x = 0$$

or $8x^6 - 9x^3 + 1 = 0$. Setting $t = x^3$ we obtain $8t^2 - 9t + 1 = 0$ which yields $t = 1, 1/8$. These give six more candidates

$$(1, \frac{1}{2}),\ (\omega, \frac{\omega^2}{2}),\ (\omega^2, \frac{\omega}{2}),\ (\frac{1}{2}, 1),\ (\frac{\omega}{2}, \omega^2),\ (\frac{\omega^2}{2}, \omega).$$

Finally, suppose

$$x^2 + xy + y^2 = 0 \ . \qquad (3)$$

From (1) minus (2) we obtain

$$8(x^4 - y^4) - 8(x^2 - y^2) + 8xy(x^2 - y^2) - 9(x - y) = 0 \ ,$$

and disregarding the possibility that $x = y$, a case already considered above, we have

$$(x+y)[8(x^2+y^2) - 8 + 8xy] = 9 \ . \qquad (4)$$

Substitution of (3) into (4) now yields

$$x + y = -\frac{9}{8} \ . \qquad (5)$$

From (5) and (3) we get

$$xy = (x+y)^2 - (x^2 + xy + y^2) = \frac{81}{64} \ . \qquad (6)$$

From (5) and (6) we see that x and y are the two roots of the equation

$$u^2 + \frac{9}{8}u + \frac{81}{64} = 0 \ ,$$

or $64u^2 + 72u + 81 = 0$. Solving, we get $u = (-9 \pm 9\sqrt{3}i)/16$. This gives two more (complex) candidates

$$(\frac{9\omega}{8}, \frac{9\omega^2}{8}),\ (\frac{9\omega^2}{8}, \frac{9\omega}{8}).$$

Straightforward substitution into (1) and (2) shows that all of these ten candidates are solutions.

Editor's note. Solutions were also received from Nicos Diamantis, student, University of Patras, Greece, and Hans Engelhaupt, Gundelsheim, Germany.

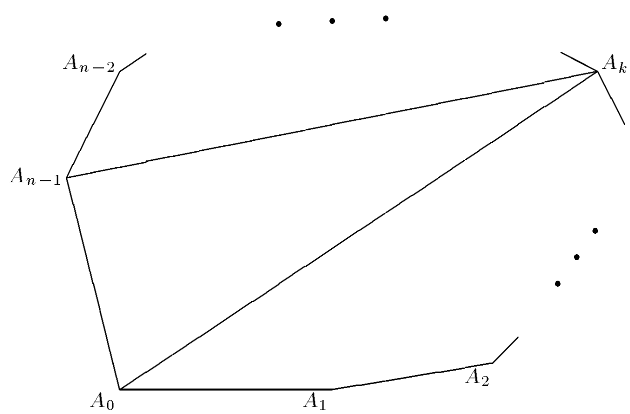
8. [1986: 20] *1985 Austria-Poland Mathematical Competition.*

The consecutive vertices of a given convex n-gon are $A_0, A_1, \ldots, A_{n-1}$. The n-gon is partitioned into $n-2$ triangles by diagonals which are non-intersecting (except possibly at the vertices). Show that there exists an enumeration $\Delta_1, \Delta_2, \ldots, \Delta_{n-2}$ of these triangles such that A_i is a vertex of Δ_i for $1 \le i \le n-2$. How many enumerations of this kind exist?

Solution by Hans Engelhaupt, Gundelsheim, Germany.

There is always just one such enumeration.

We argue by induction on n. The cases $n = 3$ and $n = 4$ are trivial. Suppose $n > 4$.



The side A_0A_{n-1} and another vertex, say A_k, $1 \le k \le n-2$, form a triangle. It is immediate that the triangle is Δ_k (since 0 and $n-2$ are not available labels). If $k = 1$ or $k = n-2$, by considering the remaining $(n-1)$-gon formed using the diagonal $A_{n-1}A_1$ or A_0A_{n-2} as appropriate and relabelling A_i as A'_{i-1} in the former case, the result follows.

So suppose $1 < k < n-2$. Now the triangle formed divides the polygon into two convex polygons $[A_0, \ldots, A_k]$ and $[A_k, \ldots, A_{n-1}]$. Also the original triangulation induces a triangulation of each of these, since the diagonals do not intersect except at endpoints. Existence of a numbering is now immediate. Uniqueness follows since the triangles $\Delta_1, \ldots, \Delta_{k-1}$ must be in $[A_0, \ldots, A_k]$ and $\Delta_{k+1}, \ldots, \Delta_{n-2}$ must be in $[A_k, \ldots, A_{n-1}]$.

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We now turn to the March 1986 numbers, and solutions to some of the problems of the *1982 Leningrad High School Olympiad (Third Round)* [1986: 39-40].

1. P_1, P_2 and P_3 are quadratic trinomials with positive leading coefficients and real roots. Show that if each pair of them has a common root, then the trinomial $P_1 + P_2 + P_3$ also has real roots. (Grade 8)

Solution by Hans Engelhaupt, Gundelsheim, Germany.

Let the trinomials be

$$P_1 : ax^2 + bx + c = 0 \text{ with } a > 0 \text{ and the real roots } u_1, u_2 \ ;$$

$$P_2 : dx^2 + ex + f = 0 \text{ with } d > 0 \text{ and the real roots } v_1, v_2 \ ;$$

and

$$P_3 : gx^2 + hx + i = 0 \text{ with } g > 0 \text{ and the real roots } w_1, w_2 \ .$$

Without loss of generality $u_1 = v_1$.

Case 1: $u_1 = w_1$ or $u_1 = w_2$.

Then $P_1 + P_2 + P_3$ has the real root u_1 so the other root is real as well.

Case 2: $u_2 = w_1$ and $v_2 = w_2$.

Then

$$P_1 + P_2 + P_3 = a(x - u_1)(x - u_2) + d(x - u_1)(x - v_2) + g(x - u_2)(x - v_2) \ (= f(x)).$$

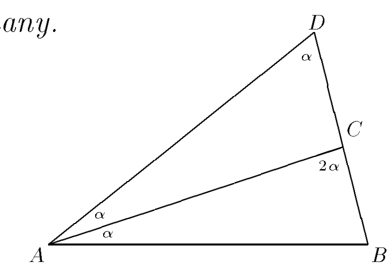
By renumbering P_1, P_2, P_3 and the roots if necessary, we may assume without loss of generality that $u_1 < u_2 < v_2$. Now $f(u_1) > 0$, $f(u_2) < 0$ and $f(v_2) > 0$. By the intermediate value theorem $f(x)$ has two real roots z_1, z_2 with

$$u_1 < z_1 < u_2 < z_2 < v_2.$$

By a suitable renumbering, any situation reduces to Case 1 or Case 2, completing the proof.

2. If in triangle ABC, $C = 2A$ and $AC = 2BC$, show that it is a right triangle. (Grade 8, 9)

Solution by Hans Engelhaupt, Gundelsheim, Germany.



Choose D on the line BC (extended) so that $CD = AC$. Then $BD = 3BC$. The triangles ADB and CAB are similar because $\angle CAD = \angle ADB = A$ $(= \alpha$, say). Thus $AB/BD = BC/AB$ and $AB^2 = BD \cdot BC = 3 \cdot BC^2$. Therefore in triangle ABC, $AB^2 + BC^2 = AC^2$, and $B = 90°$, as desired.

Editor's note. A solution using the law of sines was sent in by Bob Prielipp, University of Wisconsin–Oshkosh.

8. Prove that for any natural number k, there is an integer n such that

$$\sqrt{n + 1981^k} + \sqrt{n} = (\sqrt{1982} + 1)^k \ .$$

(Grade 9)

Solution by Bob Prielipp, University of Wisconsin–Oshkosh.
Let

$$A = \sum_{\substack{j=0 \\ j\ even}}^{k} \binom{k}{j} (\sqrt{1982})^j \ , \ B = \sum_{\substack{j=0 \\ j\ odd}}^{k} \binom{k}{j} (\sqrt{1982})^j \ .$$

[Thus A is the sum of the even-numbered terms in the expansion of $(\sqrt{1982}+1)^k$ and B is the sum of the odd-numbered terms in that expansion.] Note also that

$$(\sqrt{1982}-1)^k = \sum_{j=0}^{k} \binom{k}{j} (\sqrt{1982})^{k-j}(-1)^j = \begin{cases} B-A & \text{if } k \text{ is odd} \\ A-B & \text{if } k \text{ is even} \ . \end{cases}$$

Case 1: k is odd. Let $n = A^2$. Then

$$\begin{aligned} \sqrt{n+1981^k} + \sqrt{n} &= \sqrt{A^2 + (\sqrt{1982}-1)^k(\sqrt{1982}+1)^k} + A \\ &= \sqrt{A^2 + (B-A)(A+B)} + A \\ &= \sqrt{A^2 + B^2 - A^2} + A \\ &= B + A = (\sqrt{1982}+1)^k. \end{aligned}$$

Case 2: k is even. Let $n = B^2$. Then

$$\begin{aligned} \sqrt{n+1981^k} + \sqrt{n} &= \sqrt{B^2 + (A-B)(A+B)} + \sqrt{B^2} \\ &= \sqrt{A^2} + \sqrt{B^2} = (\sqrt{1982}+1)^k. \end{aligned}$$

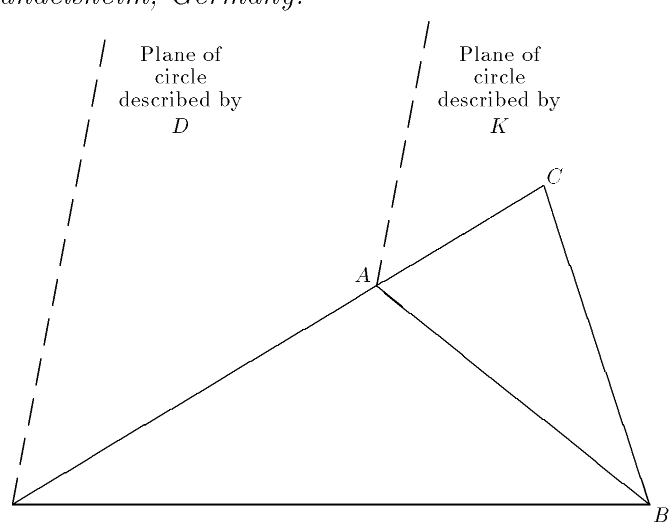
It is evident that A^2 is an integer, and B^2 is an integer since B has the form $\sqrt{1982}\ L$, for some integer L.

Editor's note. Nicos Diamantis, student, University of Patras, Greece, also solved the problem. His method was to derive that a real solution is an integer.

10. In a given tetrahedron $ABCD$, $\angle BAC + \angle BAD = 180°$. If AK is the bisector of $\angle CAD$, determine $\angle BAK$. (Grade 10)

Solution by Hans Engelhaupt, Gundelsheim, Germany.

If the triangle ABD is rotated with axis AB, the point D describes a circle in a plane orthogonal to AB. The bisector of $\angle CAD$ meets CD at K. Then K divides CD in the constant ratio AC/AD. Thus K describes a circle in a plane parallel to the plane described by D. Now A is a point of this circle [the condition $\angle BAC + \angle BAD = 180°$ means that at some stage in the rotation of ΔABD the point A will lie on the line CD]; therefore $\angle BAK = 90°$.



11. Show that it is possible to place non-zero numbers at the vertices of a given regular n-gon P so that for any set of vertices of P which are vertices of a regular k-gon $(k \leq n)$, the sum of the corresponding numbers equals zero. (Grade 10)

Solution by Nicos Diamantis, student, University of Patras, Greece.

Consider a coordinate system with $O(0,0)$ the centre of the n-gon (and therefore the centre of all regular k-gons $(k \leq n)$ which have vertices some of those of the n-gon). At the vertices of the n-gon, place the x-coordinates. [The n-gon can be rotated so that none of these x-coordinates are zero.] But we have $\sum_{i=1}^{k} \overrightarrow{OA_i} = \mathbf{0}$, where the A_i are the vertices of a regular k-gon. From this, looking at the x-coordinates we have $\sum_{i=1}^{k} x_i = 0$, where $x_1, x_2, \ldots, x_k$ are the x-coordinates of $A_1, \ldots, A_k$. This solves the problem.

*

Before turning to more recent problems, I want to apologize for leaving S.R. Cavior of the University of Buffalo off the list of solvers when I discussed problem 1 of the 24th Spanish Olympiad [1989: 67] in the January number [1991: 9]. His solution somehow had found its way into the collection for a later month.

*

For the remainder of this column, we turn to problems given in the June 1989 number of the Corner. We give solutions to all but numbers 3 and 6 of the *3rd Ibero-American Olympiad* [1989: 163–164].

1. The angles of a triangle are in arithmetical progression. The altitudes of the triangle are also in arithmetical progression. Show that the triangle is equilateral.

Solution by Bob Prielipp, University of Wisconsin–Oshkosh.

Let A, B, C be the angles of the given triangle and let h_a, h_b, h_c be the corresponding altitudes. Without loss of generality, we may assume $A \leq B \leq C$. Since the angles are in arithmetic progression $A + C = 2B$, and since $A + B + C = 180^\circ$, $B = 60^\circ$. Now also $h_c \leq h_b \leq h_a$ and $a \leq b \leq c$ where a, b, c are the side lengths opposite A, B, C, respectively.

From the law of cosines $b^2 = a^2 + c^2 - 2ac\cos 60^\circ = a^2 + c^2 - ac$. Now $2h_b = h_a + h_c$ implies that $4F/b = 2F/a + 2F/c$, where F is the area of triangle ABC, so

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} = \frac{a+c}{ac}\,, \quad \text{or} \quad b = \frac{2ac}{a+c}.$$

Since $b^2 = a^2 + c^2 - ac$, $4a^2c^2 = (a+c)^2(a^2 + c^2 - ac)$. From this we get

$$\begin{aligned} 0 &= (a+c)^2(a^2+c^2-ac) - 4a^2c^2 \\ &= [(a-c)^2 + 4ac][(a-c)^2 + ac] - 4a^2c^2 \\ &= (a-c)^4 + 5ac(a-c)^2 \\ &= (a-c)^2[(a-c)^2 + 5ac], \end{aligned}$$

and so $a = c$. This gives $a = b = c$ since $a \leq b \leq c$, and the given triangle is equilateral.

Editor's note. The problem was solved using the law of sines by Michael Selby, University of Windsor.

2. Let a, b, c, d, p and q be natural numbers different from zero such that

$$ad - bc = 1 \quad \text{and} \quad \frac{a}{b} > \frac{p}{q} > \frac{c}{d}\ .$$

Show that
(i) $q \geq b + d$;
(ii) if $q = b + d$ then $p = a + c$.

Solution by Michael Selby, University of Windsor.
Since $a/b > p/q$, $aq - pd > 0$, so $aq - pd \geq 1$. Likewise $pd - cq \geq 1$. Now

$$q = q(ad - bc) = b(pd - cq) + d(aq - pb) \geq b + d\ .$$

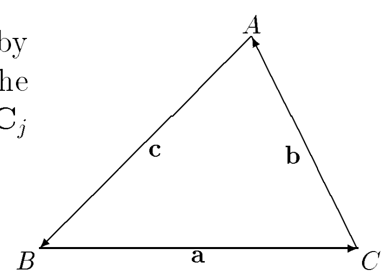
If $q = b + d$, then $b(pd - cq - 1) + d(ap - bp - 1) = 0$. Since $b > 0$, $d > 0$, $pd - cq - 1 \geq 0$ and $aq - bp - 1 \geq 0$ we must have $pd - cq = aq - pb = 1$. This yields $q(a + c) = (d + b)p$. But $q = b + d$, so $a + c = p$ as required.

4. Let ABC be a triangle with sides a, b, c. Each side is divided in n equal parts. Let S be the sum of the squares of distances from each vertex to each one of the points of subdivision of the opposite side (excepting the vertices). Show that

$$\frac{S}{a^2 + b^2 + c^2}$$

is rational.

Solution by Michael Selby, University of Windsor, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.



Consider the triangle shown with sides represented by vectors. Let $\mathbf{B}_j$ be the vector joining the vertex B to the corresponding point of subdivision on side $\mathbf{b}$, and define $\mathbf{C}_j$ and $\mathbf{A}_j$ analogously. Then

$$\mathbf{B}_j = \mathbf{a} + \frac{j}{n}\mathbf{b},$$

so

$$|\mathbf{B}_j|^2 = |\mathbf{a}|^2 + \frac{j^2}{n^2}|\mathbf{b}|^2 + \frac{2j}{n}(\mathbf{a} \cdot \mathbf{b}).$$

Therefore

$$\begin{aligned}\sum_{j=1}^{n-1} |\mathbf{B}_j|^2 &= (n-1)a^2 + b^2 \sum_{j=1}^{n-1} \frac{j^2}{n^2} + \frac{2}{n} \sum_{j=1}^{n-1} j(\mathbf{a} \cdot \mathbf{b}) \\ &= (n-1)a^2 + b^2 \frac{(n-1)(2n-1)}{6n} + (n-1)(\mathbf{a} \cdot \mathbf{b}).\end{aligned}$$

Similarly

$$\sum_{j=1}^{n-1} |\mathbf{C}_j|^2 = (n-1)b^2 + c^2 \frac{(n-1)(2n-1)}{6n} + (n-1)(\mathbf{b} \cdot \mathbf{c}),$$

$$\sum_{j=1}^{n-1} |\mathbf{A}_j|^2 = (n-1)c^2 + a^2\frac{(n-1)(2n-1)}{6n} + (n-1)(\mathbf{a}\cdot\mathbf{c}).$$

Therefore

$$S = \sum_{j=1}^{n-1}(|\mathbf{A}_j|^2 + |\mathbf{B}_j|^2 + |\mathbf{C}_j|^2) = (n-1)(a^2+b^2+c^2) + \frac{(a^2+b^2+c^2)(n-1)(2n-1)}{6n} + (n-1)(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{c}) \qquad (1)$$

Since $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$,

$$\mathbf{0} = |\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 = a^2+b^2+c^2+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}).$$

Substituting this into (1) gives

$$S = (a^2+b^2+c^2)\left(n-1+\frac{(n-1)(2n-1)}{6n} - \frac{n-1}{2}\right) .$$

Hence

$$\frac{S}{a^2+b^2+c^2} = \frac{(n-1)(5n-1)}{6n} .$$

5. We consider expressions of the form

$$x + yt + zt^2 ,$$

where $x, y, z \in \mathbf{Q}$, and $t^2 = 2$. Show that, if $x + yt + zt^2 \neq 0$, then there exist $u, v, w \in \mathbf{Q}$ such that

$$(x+yt+zt^2)(u+vt+wt^2) = 1 .$$

Solution by Michael Selby, University of Windsor.

Observe that $(x+yt+zt^2)(x+zt^2-yt) = (x+2z)^2 - 2y^2$. Now $(x+2z)^2-2y^2 \neq 0$, for otherwise $\sqrt{2} = |(x+2z)/y|$ is rational. [Note if $y = 0$ in this case, then $x + 2z = 0$ and $x + yt + zt^2 = 0$, contrary to assumption.] Let $\alpha = (x+2z)^2 - 2y^2$. Let $u = x/\alpha$, $v = -y/\alpha$ and $w = z/\alpha$. Then

$$(x+yt+zt^2)(u+vt+wt^2) = \frac{(x+2y)^2-2y^2}{\alpha} = 1 .$$

Editor's note. The result is still true if $t^3 = 2$ replaces $t^2 = 2$ in the problem.

* * *

Send me your nice solutions, and also your national and regional contests.

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **December 1, 1991,** *although solutions received after that date will also be considered until the time when a solution is published.*

1641. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Quadrilateral $ABCD$ is inscribed in circle , , with $AD < CD$. Diagonals AC and BD intersect in E, and M lies on EC so that $\angle CBM = \angle ACD$. Show that the circumcircle of ΔBME is tangent to , at B.

1642. *Proposed by Murray S. Klamkin, University of Alberta.*

Determine the maximum value of

$$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2)$$

subject to $yz + zx + xy = 1$ and $x, y, z \geq 0$.

1643. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Characterize all triangles ABC such that

$$\overline{AI_a} : \overline{BI_b} : \overline{CI_c} = \overline{BC} : \overline{CA} : \overline{AB},$$

where I_a, I_b, I_c are the excenters of ΔABC corresponding to A, B, C, respectively.

1644*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $f : \mathbf{R} \to \mathbf{R}$ be continuous such that it attains both positive and negative values, and let $n \geq 2$ be an integer. Show that there exists a strictly increasing arithmetic sequence $a_1 < \cdots < a_n$ such that $f(a_1) + \cdots + f(a_n) = 0$.

1645. *Proposed by J. Chris Fisher, University of Regina.*

Let P_1, P_2, P_3 be arbitrary points on the sides A_2A_3, A_3A_1, A_1A_2, respectively, of a triangle $A_1A_2A_3$. Let B_1 be the intersection of the perpendicular bisectors of A_1P_2 and A_1P_3, and analogously define B_2 and B_3. Prove that $\Delta B_1B_2B_3$ is similar to $\Delta A_1A_2A_3$.

1646. *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Find all positive integers n such that the polynomial

$$(a-b)^{2n}(a+b-c) + (b-c)^{2n}(b+c-a) + (c-a)^{2n}(c+a-b)$$

has $a^2 + b^2 + c^2 - ab - bc - ca$ as a factor.

1647. *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville.*

B and C are fixed points and A a variable point such that $\angle BAC$ is a fixed value. D and E are the midpoints of AB and AC respectively, and F and G are such that $FD \perp AB, GE \perp AC$, and FB and GC are perpendicular to BC. Show that $|BF| \cdot |CG|$ is independent of the location of A.

1648. *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

Evaluate $\lim_{n\to\infty}(u_n/\sqrt{n})$, where $\{u_n\}$ is defined by $u_0 = u_1 = u_2 = 1$ and

$$u_{n+3} = u_{n+2} + \frac{u_n}{2n+6}\ , \quad n = 0, 1, \ldots\ .$$

1649*. *Proposed by D.M. Milošević, Pranjani, Yugoslavia.*

Prove or disprove that

$$\sum \cot\frac{\alpha}{2}\ -\ 2\sum \cot\alpha \ge \sqrt{3},$$

where the sums are cyclic over the angles α, β, γ of a triangle.

1650. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Find all real numbers α for which the equality

$$[\sqrt{n+\alpha} + \sqrt{n}] = [\sqrt{4n+1}]$$

holds for every positive integer n. Here [ ] denotes the greatest integer function. (This problem was inspired by problem 5 of the 1987 Canadian Mathematics Olympiad [1987: 214].)

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

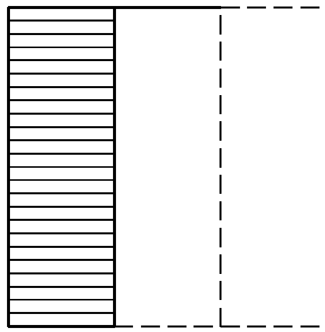
875*. [1983: 241; 1984: 338] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

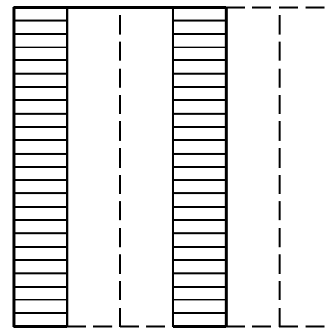
Can a square be dissected into three congruent nonrectangular pieces?

II. *Solution by Sam Maltby, student, University of Calgary.*

We show that the answer is no.

Let the unit square $ABCD$ be cut into three congruent pieces P_1, P_2 and P_3, and suppose these pieces are not rectangles. Here we will assume that our "pieces" contain their boundaries, and therefore that two pieces are allowed to overlap on their boundaries but not elsewhere. We must also assume that the pieces are connected, and moreover that they contain no "isthmuses" or "tails"; otherwise it is easy to find counterexamples, as suggested by the following figures.





The effect of these assumptions is that we take the pieces to have the following three properties:

(1) If a piece contains a vertex of the square but does not contain at least part of both edges at that vertex, then some other piece must also contain that vertex;

(2) A segment of the boundary of the square cannot belong to more than one piece;

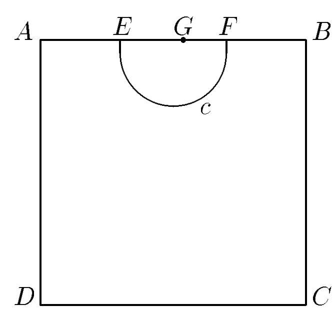
(3) Between any two points X and Y of a piece P there is a curve (called an *incurve* in P between X and Y) with X and Y as endpoints such that all points of the curve other than X and Y are interior points of P. An incurve of one piece cannot intersect another piece except possibly at its endpoints.

Note that since the pieces are congruent it must be possible through some combination of translation, rotation or reflection to place one piece on another. This will take each curve in the first piece to some curve in the second; we shall say that these two curves *correspond*.

We now give a series of lemmas and eventually arrive at the desired conclusion.

Lemma 1. The intersection of a piece with a side of the square is connected.

Proof. Suppose for a contradiction that the intersection of one of the pieces, say P_1, with one of the sides of the square, say AB, is not connected. Then there are two points E and F on AB in P_1 such that EF is not entirely in P_1; say that point G between E and F is in P_2. Let c be an incurve in P_1 between E and F. If there is some point H in P_2 on the other side of c from G, then an incurve in P_2 between G and H must intersect c, but it cannot have a common endpoint with c. This contradicts (3), so H cannot exist. Thus P_2 is enclosed by c and EF. But then the convex hull of P_1 is larger than the convex hull of P_2, which contradicts P_1 and P_2 being congruent. □

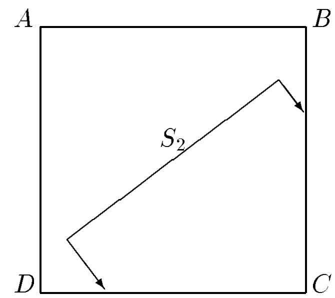


Lemma 2. No piece contains two opposite corners of the square.

Proof. Suppose that, for instance, P_1 contains A and C. Then clearly P_2 and P_3 must each also contain opposite corners of the square. If P_2 contains B and D, then there is an incurve in P_1 between A and C and one in P_2 between B and D. These must intersect, but they do not have a common endpoint, which contradicts (3). Thus P_2 and likewise P_3 must contain A and C.

Now some piece, say P_1, contains B. Then P_2 and P_3 must each contain a point at distance 1 from both A and C. But B and D are the only such points, so two of the pieces (say P_1 and P_2) contain, say, B. By Lemma 1, both P_1 and P_2 contain BC, which is impossible by (2). □

By the Pigeonhole Principle, one of the three pieces, say P_1, contains at least two corners of the square, and by Lemma 2 these corners must be adjacent, say A and B. By Lemma 1 P_1 contains the entire side AB. Then each of P_2 and P_3 contains a segment which corresponds to AB; call these segments S_2 and S_3 respectively. It follows that



(4) P_2 is contained in the area enclosed by S_2 and two rays with endpoints at the ends of S_2 and which are perpendicular to S_2; and likewise for P_3 and S_3.

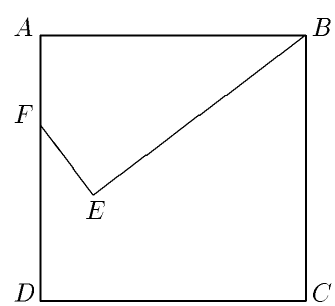
Lemma 3. S_2 (and similarly S_3) cannot be parallel to BC.

Proof. Suppose S_2 is parallel to BC. If it is not BC or AD then there is an incurve in P_2 between the endpoints of S_2 which intersects one in P_1 between A and B. This contradicts (3), so S_2 must be BC or AD, say BC. Then since AB and BC correspond, P_2 must be either the reflection of P_1 through BD or the rotation of P_1 through 90° about the centre of the square.

In the former case, P_3 must be symmetric across BD, so P_1 must also be symmetric through some axis. Since the reflection cannot take AB outside the square, the axis must be (i) AC, (ii) the perpendicular bisector of AB, (iii) a line parallel to AB, or (iv) a line through either A or B which makes an angle less than 45° with AB.

If (i), then P_1 contains B and D, which contradicts Lemma 2. If (ii), then no point on AD other than A can be in P_1; for by Lemma 1 some line segment in AD would be in P_1, and thus (by reflection) a segment in BC would be in P_2 as well as P_1, which is impossible by (2). Since A cannot be in P_2 by Lemma 2, by (1) A is in P_3. By the symmetry of P_3, C must also be in P_3, which contradicts Lemma 2. If (iii), the reflection takes B to some other point on BC, so P_1 contains some segment of BC which is also in P_2, again a contradiction.

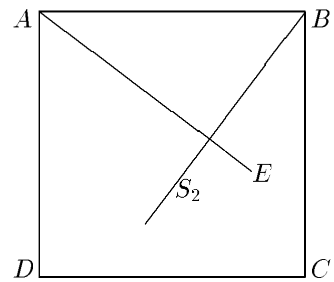
Finally, suppose (iv). If the axis is through A then P_1 contains no point of AD other than A, so P_3 contains A and (by reflection) C, contrary to Lemma 2 as before. So the axis is through B; let E be the reflection of A through the axis. Since P_1 is bounded by AD, which is perpendicular to AB, it is also bounded by the perpendicular to BE at E. Let F be the intersection of the perpendicular and AD. Then P_3 contains DF and by reflection a segment of equal length along DC. Now $\angle ABF \le 22.5°$, so $|AF| \le \tan 22.5° = \sqrt{2}-1$, so $|DF| = 1-|AF| \ge 2-\sqrt{2}$. Then P_1 must have two segments of length at least $2-\sqrt{2}$ in its boundary which are at right angles (corresponding to DF and its reflection) and so that one is the image of the other across the axis of reflection BF. It is easy to see that this is impossible. Therefore P_2 is not a reflection of P_1.



If P_2 is a rotation of P_1 then P_1 cannot contain any line segment in AD, since by the rotation P_2 would contain some line segment in AB which would also be in P_1, contrary to (2). As before, P_2 cannot contain A and so A is in P_3 by (1). Likewise C is also in P_3, which again contradicts Lemma 2. □

Lemma 4. P_2 cannot contain a side of the square unless that side is parallel to S_2. Likewise for P_3 and S_3.

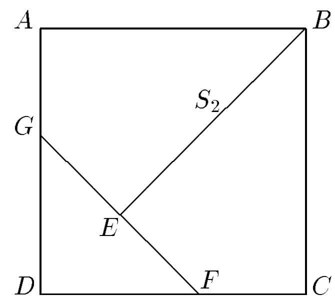
Proof. First suppose that P_2 contains BC. By Lemma 3 BC is not parallel to S_2, so by (4) S_2 must have either B or C as one of its endpoints. There are now four cases.
Case (a). B is an endpoint of S_2 and B in P_2 corresponds to A in P_1.



There is a segment AE in P_1 which corresponds to BC in P_2 such that $\angle EAB$ is equal to the angle between S_2 and BC. If AE and S_2 don't intersect, then, extending them until they meet, we get a right triangle with hypotenuse AB of length 1 and one side of length at least 1, which is impossible. So they must intersect at some point interior to both, which is impossible by (3).
Case (b). C is an endpoint of S_2 and C in P_2 corresponds to A in P_1.

There must be a segment AE in P_1 which corresponds to BC in P_2. Then $\angle EAB$ is equal to the angle made by S_2 and BC; but then since S_2 is not BC or CD we see that S_2 and AE intersect at some point other than A or E, which is impossible.
Case (c). C is an endpoint of S_2 and C in P_2 corresponds to B in P_1.

The argument here is essentially the same as that in case (a).
Case (d). B is an endpoint of S_2 and B in P_2 corresponds to B in P_1.



Let E be the other endpoint of S_2. Note that by Lemma 2 neither P_1 nor P_2 contains D, so P_3 must, and so P_3 does not contain B. Then S_2 must bisect the right angle ABC, and P_2 must be a 45° rotation of P_1 around B. Since P_2 is bounded by S_2, BC, and the perpendicular to S_2 through E, P_3 must contain the segment DF along DC with length $2-\sqrt{2}$. Since P_2 is contained in $BEFC$, by rotation P_1 is contained in $ABEG$, so P_3 also contains the segment DG of length $2-\sqrt{2}$, where EF meets AD at G.

Now S_3 is not parallel to BC by Lemma 3, and it cannot be parallel to CD because if it were it would have to be CD itself, but then we could replace AB by CD and P_1 by P_3 in case (b) or (c) to get the result. So S_3 makes an acute angle with CD. By applying (4) to P_3 we see that G, D and F must all be on the same side of S_3, but S_3 cannot intersect the interior of EB since EB lies on the boundary of P_1 and P_2. Thus S_3 must be GF; but $|GF| = 2(\sqrt{2}-1) < 1$, a contradiction. This finishes the proof that P_2 (and P_3) cannot contain BC. Similarly, P_2 and P_3 cannot contain AD.

Now suppose that P_2 contains CD but S_2 is not parallel to CD. By (4) S_2 must have either C or D as an endpoint, say C. S_2 cannot be perpendicular to CD since it would then be BC contrary to Lemma 3. But since P_1 does not contain C, P_3 contains C by (1). Now C in P_2 cannot correspond to A in P_1, since then P_1 would contain no point of AD other than A, and A cannot be in P_2, so P_3 would contain A and C contrary to Lemma 2. So C in P_2 corresponds to B in P_1. But then P_3 contains B and thus BC, which is impossible by the above. □

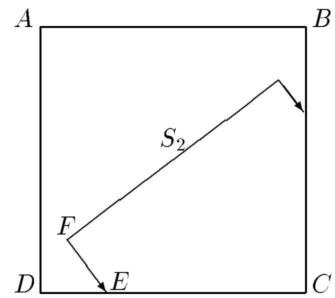
Lemma 5. One of P_2 and P_3 must contain a side of the square.

Proof. To obtain a contradiction suppose that neither P_2 nor P_3 contains a side of the square. Then S_2 and S_3 are not sides of the square and so must be inside the square.

First note that P_1 contains neither C nor D by Lemma 2, and neither P_2 nor P_3 contains both since it would then contain CD. Thus one, say P_2, contains C and the other (P_3) contains D.

Also note that P_1 contains some point of AD other than A since otherwise P_3 would contain the entire side AD. Likewise P_1 contains some point of BC. By congruence then, there are right angles in P_2 (P_3) at both ends of S_2 (S_3).

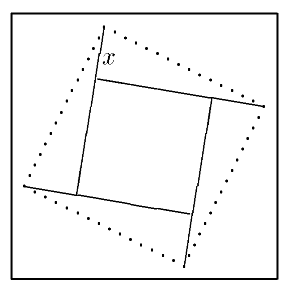
Now S_2 is not parallel to CD because if it were then P_1 would be a rectangle, contrary to assumption. Neither is it perpendicular, by Lemma 3. Thus by (4) S_2 must be oriented as shown. But then C is the unique point in P_2 furthest from S_2. Thus there is one point of P_1 at maximum distance from AB and so P_1 cannot contain more than one point of CD. Then P_2 and P_3 have a common point E on CD, where we assume without loss of generality that E is no farther from D than from C. By the orientation of S_2 at least one end, call it F, of S_2 must be strictly closer to AD than E is. (Note $E \neq F$.)



Since (as above) D is the only point in P_3 at maximum distance from S_3, D in P_3 corresponds to C in P_2. Thus P_2 is either a reflection of P_3 through the perpendicular bisector of CD or a rotation of P_3 through 90° about the centre of the square.

If it is a reflection then no interior point of P_2 or of P_3 is on the perpendicular bisector of CD since it would then belong to both pieces. But there is an incurve from F to C, and since F is closer to AD than E is this curve must cross the bisector. This is impossible, so P_2 is not a reflection of P_3.

So P_2 must be a rotation of P_3 through 90° about the centre of the square. Then S_2 is perpendicular to S_3, and since neither S_2 nor S_3 are edges of the square it follows that S_2 and S_3 must cross at a point interior to both. (Otherwise we can rotate both lines by 180° and extend the resulting four lines until they meet, as in the diagram. Let x be the length of the segment past the point of intersection. The convex hull of this figure is a square of side at least $\sqrt{1+x^2}$, and so has area at least $1+x^2$, but is contained in the unit square. Thus x must be 0, which means that the original lines were sides of the square.) Since S_2 and S_3 are on the boundaries of P_2 and P_3 respectively, this is impossible. □



Now by Lemma 5, P_2 (say) contains a side of the square, but by Lemmas 3 and 4 it does not contain BC or AD, so it must contain CD. By Lemma 4 S_2 is parallel to CD. If CD is not S_2 then P_2 is a rectangle; therefore CD is S_2 and then P_1 is either a reflection of P_2 through the perpendicular bisector of BC or a rotation of P_2 through 180° around the centre of the square.

In the case of a reflection, P_3 must be symmetric through the perpendicular bisector of BC. However P_3 cannot contain BC or AD; therefore P_1 contains some points in AD and BC other than A and B, and so S_3 must have the perpendiculars from its endpoints at least partly in P_3. By (4) and the symmetry of P_3, either (i) S_3 is at $45°$ to the perpendicular bisector of BC with an endpoint on it, or (ii) S_3 is parallel to AB (it cannot be perpendicular to AB by Lemma 3). However if (i) then since S_3 has length 1 it passes outside the square, and if (ii) then either P_1 or P_2 is a rectangle, which means we have a rectangular dissection.

If P_1 is a rotation of P_2, then rotating the entire square takes S_3 to another side of length one parallel to S_3 which is also in P_3. Then P_1 must have a side of length one parallel to AB and is easily seen to be rectangular, so the dissection is again rectangular.

The result has now been proved.

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1523. [1990: 74] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $0 < t \leq 1/2$ be fixed. Show that

$$\sum \cos tA \geq 2 + \sqrt{2}\cos(t + 1/4)\pi + \sum \sin tA,$$

where the sums are cyclic over the angles A, B, C of a triangle. [This generalizes Murray Klamkin's problem E3180 in the *Amer. Math. Monthly* (solution p. 771, October 1988).]

I. *Solution by Seung-Jin Bang, Seoul, Republic of Korea.*

Our solution is essentially the same as O.P. Lossers in the October 1988 *Monthly.* Let

$$V(A,B,C,t) = \sum \cos tA - \sum \sin tA.$$

Then we see that

$$\begin{aligned} V(A,B,C,t) &= \cos tA + \cos tB - (\sin tA + \sin tB) + \cos tC - \sin tC \\ &= 2\cos\frac{t(A-B)}{2}\left(\cos\frac{t(A+B)}{2} - \sin\frac{t(A+B)}{2}\right) + \cos tC - \sin tC \\ &= 2\cos\frac{t(A-B)}{2}\left(\cos\frac{t(\pi-C)}{2} - \sin\frac{t(\pi-C)}{2}\right) + \cos tC - \sin tC. \end{aligned}$$

Note that

$$\begin{aligned} V(A,B,0,t) &= 2\sqrt{2}\cos\frac{t(A-B)}{2}\cos\left(\frac{t\pi}{2}+\frac{\pi}{4}\right) + 1 \\ &\geq 2\sqrt{2}\cos\frac{t\pi}{2}\cos\left(\frac{t\pi}{2}+\frac{\pi}{4}\right) + 1 \\ &= \sqrt{2}\cos(\pi t + \frac{\pi}{4}) + 2. \end{aligned}$$

For fixed $C \neq 0$, $V(A,B,C,t)$ is minimized only when $A = 0$ or $B = 0$, and maximized only when $A = B$, because

$$\left|\frac{t(A+B)}{2}\right| \leq \frac{\pi}{4} \quad \Rightarrow \quad \cos\frac{t(A+B)}{2} > \sin\frac{t(A+B)}{2}.$$

By symmetry, $V(A,B,C,t)$ is minimized on the boundary $A \cdot B \cdot C = 0$ [in fact when two of A, B, C are 0], and maximized only at $A = B = C = \pi/3$. We conclude that

$$2 + \sqrt{2}\cos(\pi t + \frac{\pi}{4}) \leq V(A,B,C,t) \leq 3\sqrt{2}\cos(\frac{\pi t}{3} + \frac{\pi}{4}).$$

The left-hand inequality answers the problem.

II. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

Slightly more generally, we assert that

$$\sum_{i=1}^{n}(\cos x_i - \sin x_i) \geq (n-1) + \sqrt{2}\cos\left(\frac{\pi}{4} + \sum_{i=1}^{n} x_i\right) \tag{1}$$

for $x_1, \ldots, x_n \geq 0$, $\sum_{i=1}^{n} x_i \leq \pi/2$. For $n = 3$, $x_1 = tA, x_2 = tB, x_3 = tC$, this is the inequality of the problem.

Defining

$$f(x) = 1 + \sin x - \cos x = 1 - \sqrt{2}\cos(\frac{\pi}{4} + x),$$

we rewrite (1) as

$$f\left(\sum_{i=1}^{n} x_i\right) \geq \sum_{i=1}^{n} f(x_i), \tag{2}$$

for $x_1, \ldots, x_n \geq 0$, $\sum_{i=1}^{n} x_i \leq \pi/2$. (This is a yet more natural form of the statement.)

It suffices to prove (2) for $n = 2$; obvious induction then does the rest. And for $n = 2$, writing x and y for x_1 and x_2, we have

$$\begin{aligned} f(x+y) - f(x) - f(y) &= (\cos y - \cos(x+y)) - (1 - \cos x) + (\sin(x+y) - \sin y) - \sin x \\ &= \left(2\sin(y + \frac{x}{2})\sin\frac{x}{2} - 2\sin^2\frac{x}{2}\right) + \left(2\cos(y + \frac{x}{2})\sin\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}\right) \\ &= 2\sin\frac{x}{2}\left(2\cos\frac{x+y}{2}\sin\frac{y}{2} - 2\sin\frac{x+y}{2}\sin\frac{y}{2}\right) \\ &= 4\sin\frac{x}{2}\sin\frac{y}{2}\left(\cos\frac{x+y}{2} - \sin\frac{x+y}{2}\right) \geq 0 \end{aligned}$$

because $(x+y)/2 \in [0, \pi/4]$.

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; and the proposer. There was one incorrect solution submitted.

The problem was also proposed independently (without solution) by Robert E. Shafer, Berkeley, California.

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1524. [1990:74] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

ABC is a triangle with sides a, b, c and area F, and P is an interior point. Put $R_1 = AP$, $R_2 = BP$, $R_3 = CP$. Prove that the triangle with sides aR_1, bR_2, cR_3 has circumradius at least $4F/(3\sqrt{3})$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

In any triangle,

$$R \geq \frac{1}{3\sqrt{3}}(a+b+c) \tag{1}$$

([1], item 5.3). Thus, using item 12.19 of [1], we get for the circumradius $\tilde{R}$ under consideration:

$$\begin{aligned}\tilde{R} &\geq \frac{1}{3\sqrt{3}}(aR_1 + bR_2 + cR_3) \geq \frac{2}{3\sqrt{3}}(ar_1 + br_2 + cr_3) \\ &= \frac{2}{3\sqrt{3}}(2F) = \frac{4F}{3\sqrt{3}}\ .\end{aligned}$$

Reference:

[1] Bottema et al, *Geometric Inequalities.*

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; and the proposer.

Klamkin and Kuczma note that inequality (1) *is equivalent to the fact that the perimeter of a triangle inscribed in a circle is maximized when the triangle is equilateral.*

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1525. [1990: 75] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Let m, n be given positive integers and d be their greatest common divisor. Let $x = 2^m - 1$, $y = 2^n + 1$.

(a) If m/d is odd, prove that x and y are coprime.

(b) Determine the greatest common divisor of x and y when m/d is even.

Solution by Kenneth M. Wilke, Topeka, Kansas.

Putting $m_1 = m/d$ and $n_1 = n/d$, we have $(m_1, n_1) = 1$. Let $\delta = (x, y)$ and put $k = x/\delta$, $\ell = y/\delta$. We want to find δ.

(a) If $m_1 = m/d$ is odd, then

$$(k\delta + 1)^{n_1} = 2^{mn_1} = 2^{m_1n_1d} = 2^{nm_1} = (\ell\delta - 1)^{m_1}.$$

But by the binomial theorem, $(k\delta+1)^{n_1} = K\delta+1$ for some integer K, and $(\ell\delta-1)^{m_1} = L\delta-1$ for some integer L since m_1 is odd. Hence $K\delta + 1 = L\delta - 1$, or $2 = \delta(L - K)$. Thus $\delta|2$ and since both x and y are odd, δ must be odd also. Hence $\delta = 1$, as required.

(b) If $m_1 = m/d$ is even, say $m_1 = 2m_2$, then $(m_2, n_1) = 1$. We shall use the known result that for natural numbers a, m, n such that $a > 1$, $(a^m - 1, a^n - 1) = a^{(m,n)} - 1$. Let $y' = 2^n - 1$. Then $\delta | yy' = 2^{2n} - 1$, so $\delta | (x, yy')$. But

$$\begin{aligned}(x, yy') &= (2^{2m_2d} - 1, 2^{2n_1d} - 1) = 2^{(2m_2d, 2n_1d)} - 1 \\ &= 2^{2d} - 1 = (2^d + 1)(2^d - 1)\ .\end{aligned}$$

Since $(m_1, n_1) = 1$ and m_1 is even, n_1 must be odd. Hence $(2^d + 1) | (2^{n_1d} + 1) = y$. Also, if $r > 1$ is any divisor of $2^d - 1$, we have

$$\frac{y}{r} = \frac{2^{n_1d} + 1}{r} = \frac{2^{n_1d} - 1}{r} + \frac{2}{r}\ ,$$

where $r | (2^{n_1d} - 1)$, and hence y/r is not an integer. Thus we must have $\delta = (x, y) = 2^d + 1$ in this case.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; and the proposer.

Bang, Janous and Lau note that the problem is a special case of problem E3288 of the American Math. Monthly, *solution on pp. 344-345 of Volume 97 (1990).*

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1526. [1990: 75] *Proposed by Sydney Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let n and q denote positive integers. The identity

$$\sum_{k=1}^{n} k \binom{n}{k} q^{n-k} = n(q+1)^{n-1}$$

can be proved easily from the Binomial Theorem. Establish this identity by a combinatorial argument.

Solution by H.L. Abbott, University of Alberta.

The Mathematics Department at a certain university has n members. The administration of the department is handled by an executive committee whose chairman also serves as chairman of the department. There are no restrictions, except the obvious ones, on the size of the executive committee. For example, during those years when many onerous problems are expected to arise it may be a committee of one, while in times when little of any consequence needs attention it may consist of the whole department. Each member of the department who is not a member of the executive committee is required to serve on exactly one of q committees. There is no restriction on the size of these committees either. Indeed, some of them need not have any members at all. This, for example, will be the case when the size k of the executive committee is such that $q + k > n$.

Late one evening as the chairman was leaving the department he remarked to his secretary that there is a simple expression for the number of possible administrative structures for the department. "Observe," he said, "that the number of ways of choosing an executive committee of size k is $\binom{n}{k}$. The chairman of this committee, and thus of the department, may then be chosen in k ways and the remaining $n-k$ members of the department may then be assigned their tasks in q^{n-k} ways. Thus the number of possible bureaucracies is $\sum_{k=1}^{n} k\binom{n}{k}q^{n-k}$."

His secretary, almost without hesitation, replied, "Surely there is a much simpler expression for this number. The chairman of the executive committee may be chosen in n ways and after this choice has been made the remaining $n-1$ members of the department may be assigned their administrative chores in $(q+1)^{n-1}$ ways. Thus the number is $n(q+1)^{n-1}$."

A few moments later the chairman related this conversation to the caretaker as they rode the elevator to the first floor. "It shows," said the caretaker, "that your secretary is the one who counts."

A student who was in the elevator was overjoyed upon hearing this discussion. She realized that she could now solve the one remaining question on her combinatorics assignment which was due at the next class. The problem called for a combinatorial proof that

$$\sum_{k=2}^{n} k(k-1)\binom{n}{k}q^{n-k} = n(n-1)(q+1)^{n-2}.$$

As soon as she arrived home she wrote out the solution: The Mathematics Department at a certain university has n members. The administration of the department is handled by an executive committee whose chairman and associate chairman also serve as chairman and associate chairman of the department.... .

Also solved by JACQUES CHONÉ, Clermont-Ferrand, France; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOM LEINSTER, Lancing College, England; and the proposers.

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1527. [1990: 75] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

In quadrilateral $ABCD$ the midpoints of AB, BC, CD and DA are P, Q, R and S respectively. T is the intersection point of AC and BD, M that of PR and QS. G is the centre of gravity of $ABCD$. Show that T, M and G are collinear, and that $\overline{TM} : \overline{MG} = 3 : 1$.

Solution by John Rausen, New York.

The "center of gravity" of a quadrilateral is ambiguous. For a triangle, the centroid G (point of intersection of the medians) is a "center of gravity" in two different ways: (1) G is the center of mass of a system of three equal point-masses at the vertices; (2) G is also the center of mass of a uniform mass distribution on a thin plate ("lamina") covering the triangle. In the case of a (plane) quadrilateral, we get two different points: (1) the center of mass of a set of four equal masses, say unit masses, at the vertices. This is point

M of the statement because, by an elementary principle of mechanics, we can replace the unit masses at A and B by a mass of 2 units at the midpoint P of AB, and similarly replace the unit masses at C, D by a mass of 2 at point R; then the center of mass of the system is the midpoint of PR. But it is also the midpoint of QS, hence it is point $M = PR \cap QS$. Note that, by the same reasoning, M is also the midpoint of the line segment connecting the midpoints U, V of the diagonals AC, BD. M is often called the centroid of the quadrilateral $ABCD$ [1].

Therefore the point G of the problem must be (2) the center of mass of a uniform distribution over the surface of the quadrilateral (and the problem provides an interesting relation between the two "centers of gravity"). Point G can be located by the same mechanical principle. Assuming first that $ABCD$ is convex, suppose the mass of triangle BCD is concentrated at its centroid A', and the mass of triangle BAD at its centroid C'. Then, since the quadrilateral is the union of these two triangles (with no overlap), the center of mass G is some point on line $A'C'$ (the exact position determined by the ratio of the areas of the two triangles). But by the same reasoning, G is on line $B'D'$, where B', D' are the centroids, respectively, of triangles ADC, ABC. Therefore point G is the intersection of lines $A'C'$ and $B'D'$.

If the quadrilateral is not convex, in one case instead of the quadrilateral being the union of two triangles, we would have one of the triangles the union of the quadrilateral and the other triangle, but then the three centers of mass are still collinear, so the conclusion $G = A'C' \cap B'D'$ holds in all cases.

Returning to point M, it can also be obtained by first combining the unit masses at points B, C, D into a mass of 3 units at the centroid A' of triangle BCD. Then M is the point on line AA' such that $\overrightarrow{AM} = 3\,\overrightarrow{MA'}$, or $\overrightarrow{MA'} = -\frac{1}{3}\,\overrightarrow{MA}$. Similarly, $\overrightarrow{MB'} = -\frac{1}{3}\,\overrightarrow{MB}$, $\overrightarrow{MC'} = -\frac{1}{3}\,\overrightarrow{MC}$ and $\overrightarrow{MD'} = -\frac{1}{3}\,\overrightarrow{MD}$. Therefore quadrilateral $A'B'C'D'$ is the image of quadrilateral $ABCD$ under the homothetic transformation with center M and ratio $-1/3$, i.e., the point transformation which takes any point P in the plane into point P' defined by $\overrightarrow{MP'} = -\frac{1}{3}\,\overrightarrow{MP}$. Under such a transformation, any point associated to $ABCD$ goes into the corresponding point associated to $A'B'C'D'$. Thus, since M goes into itself, it is also the centroid of $A'B'C'D'$. As to point $T = AC \cap BD$, its image is the point $A'C' \cap B'D' = G$, and it is true that $\overrightarrow{MG} = -\frac{1}{3}\,\overrightarrow{MT}$, or $\overrightarrow{TM} = 3\,\overrightarrow{MG}$, as required.

Reference:
[1] Nathan Altshiller-Court, *College Geometry*, 2nd edition, 1952, New York.

Also solved by WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; TOM LEINSTER, Lancing College, England; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

Lau located the problem as a worked example in §7.12 *of Humphrey's* Intermediate Mechanics Vol. 2 (Statics), *Second Edition, Longmans, 1964. (On p. 237-238 of* §151 *in* *the 1961 edition.) The proposer took the problem from* Journal de Math. Élémentaires*, April 1917, no. 8477.*

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1528*. [1990: 75] *Proposed by Ji Chen, Ningbo University, China.*

If a, b, c, d are positive real numbers such that $a + b + c + d = 2$, prove or disprove that

$$\frac{a^2}{(a^2+1)^2} + \frac{b^2}{(b^2+1)^2} + \frac{c^2}{(c^2+1)^2} + \frac{d^2}{(d^2+1)^2} \le \frac{16}{25}.$$

Solution by G.P. Henderson, Campbellcroft, Ontario.

We will prove that the inequality is true.

Set

$$f(x) = \frac{x^2}{(x^2+1)^2} \ .$$

Then

$$f'(x) = \frac{2x(1-x^2)}{(x^2+1)^3} \ , \qquad f''(x) = \frac{2(3x^4 - 8x^2 + 1)}{(x^2+1)^4} \ .$$

f has a minimum at $x = 0$, a maximum at $x = 1$, then decreases and approaches zero as $x \to \infty$. There is a point of inflection at $x = \sqrt{(4-\sqrt{13})/3} \approx 0.36$.

The tangent at $x = 1/2$ is

$$y = \frac{4}{25} + \frac{48}{125}\left(x - \frac{1}{2}\right) \ .$$

Since $f''(1/2) < 0$, the curve is below the tangent near $x = 1/2$. At $x = 0$, it is above the tangent. They intersect at x_1, the real root of

$$12x^3 + 11x^2 + 32x - 4 = 0 \ .$$

The polynomial is negative at $x = 0$ and positive at $x = 1/8$. Therefore $x_1 < 1/8$.

Set

$$F = f(a) + f(b) + f(c) + f(d) \ , \quad 0 \le a, b, c, d, \quad \sum a = 2 \ .$$

If $a, b, c, d \ge x_1$,

$$F \le \sum \left[\frac{4}{25} + \frac{48}{125}\left(a - \frac{1}{2}\right)\right] = \frac{1}{125}\sum(48a - 4) = \frac{48}{125}\cdot 2 - \frac{4}{125}\cdot 4 = \frac{16}{25} \ ,$$

as claimed.

Suppose now, that at least one of $a, b, c, d < x_1$, say $a < x_1$. Set $t = (2-a)/3$. Since $0 \le a < 1/8$, $5/8 < t \le 2/3$.

For a fixed, we will show that the maximum F occurs at $b = c = d = t$. The tangent at $x = t$ has the form

$$y = f(t) + m(x - t) \ . \qquad (1)$$

At $x=t$, the curve is below the tangent because t is greater than the abscissa of the point of inflection. It is easily verified that the tangent at $x=1/\sqrt{3}$ passes through the origin. Since $t>1/\sqrt{3}$, (1) is still above the curve at $x=0$. Therefore

$$f(x) \le f(t) + m(x-t)\ .$$

Using this for $x=b,c,d$,

$$F \le f(a) + 3f(t) + m(b+c+d-3t) = f(a) + 3f(t)\ .$$

It remains to show that for $t=(2-a)/3$,

$$G(a) = f(a) + 3f(t) < 16/25\ .$$

We have

$$\begin{aligned} G'(a) &= f'(a) - f'(t) \le \max_{0\le a\le \frac{1}{8}} f'(a) \ - \min_{\frac{5}{8}\le t\le \frac{2}{3}} f'(t) \\ &= f'(1/8) - f'(2/3) = \frac{63\cdot 64\cdot 16}{65^3} - \frac{20\cdot 27}{13^3} < 0\ . \end{aligned}$$

Therefore

$$G(a) \le G(0) = \frac{108}{169} < \frac{16}{25}\ .$$

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; and JOHN LINDSEY, Northern Illinois University, Dekalb.

Equality of course holds for $a=b=c=d$. Engelhaupt notes that (with $f(x)$ defined as in Henderson's solution) it is **not always true**, *for positive a,b,c,d satisfying $a+b+c+d=k$, that*

$$f(a)+f(b)+f(c)+f(d) \le 4f(k/4)\ . \qquad (2)$$

For example, when $k=1$, he obtains

$$f(0.2)+f(0.2)+f(0.3)+f(0.3) > 4f(0.25)\ ,$$

and when $k=8$, he obtains

$$f(1.8)+f(1.8)+f(2.2)+f(2.2) > 4f(2)\ .$$

He asks for which values of k (2) is true.

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1530*. [1990: 75] *Proposed by D.S. Mitrinović, University of Belgrade, and J.E. Pečarić, University of Zagreb.*

Let

$$I_k = \frac{\int_0^{\pi/2} \sin^{2k} x\,dx}{\int_0^{\pi/2} \sin^{2k+1} x\,dx}$$

where k is a natural number. Prove that

$$1 \le I_k \le 1 + \frac{1}{2k}.$$

Solution by Marcin E. Kuczma, Warszawa, Poland.

This is a variation on a theme of Wallis. The infinite product formula

$$1 = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\ldots\cdot\frac{2k-1}{2k}\cdot\frac{2k+1}{2k}\cdot\ldots \tag{1}$$

is a lesson we've all been taught in the course of elementary calculus. The usual way it's derived in textbooks is by considering the integrals

$$c_n = \int_0^{\pi/2} \sin^n x\,dx$$

and their basic recursion (resulting from integration by parts)

$$c_n = \frac{n-1}{n}c_{n-2} \qquad (c_0 = \pi/2\ , \quad c_1 = 1)\ .$$

For I_k this yields the recursion

$$I_k = \frac{c_{2k}}{c_{2k+1}} = \left(\frac{2k-1}{2k}c_{2k-2}\right)\left(\frac{2k}{2k+1}c_{2k-1}\right)^{-1} = \frac{2k-1}{2k}\cdot\frac{2k+1}{2k}I_{k-1} \tag{2}$$

(with $I_0 = \pi/2$), and hence we get for $k = 1, 2, 3, \ldots$

$$I_k = \frac{\pi}{2}\left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{3}{4}\cdot\frac{5}{4}\right)\ldots\left(\frac{2k-1}{2k}\cdot\frac{2k+1}{2k}\right)\ . \tag{3}$$

This is just a piece of (1). If we however truncate (1) one step earlier, we obtain another partial product of (1), which we denote J_k. Thus

$$J_k = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot\frac{2k-1}{2k}\ , \quad J_{k+1} = J_k\cdot\frac{2k+1}{2k}\cdot\frac{2k+1}{2k+2}\ . \tag{4}$$

Relations (2) and (4) show that $< I_k >$ is a decreasing sequence, $< J_k >$ is increasing, and both converge to 1, in agreement with (1). So we have for each k

$$I_k > 1 > J_k = \frac{2k}{2k+1}I_k\ ,$$

and this is exactly what we had to show.

With a little further effort we can obtain a much more precise two-sided estimate for I_k. By (1) and (3),

$$I_k = \prod_{j=k+1}^{\infty} \left(\frac{2j}{2j-1} \cdot \frac{2j}{2j+1} \right) . \tag{5}$$

It follows from the Lagrange (intermediate value) theorem that

$$\frac{1}{x} < \ln x - \ln(x-1) < \frac{1}{x-1} \quad \text{for } x > 1,$$

which with $x = 4j^2$ gives

$$\frac{1}{4j^2} < \ln \frac{4j^2}{4j^2-1} < \frac{1}{4j^2-1} .$$

Thus, by (5),

$$\ln I_k = \sum_{j=k+1}^{\infty} \ln \left(\frac{4j^2}{4j^2-1} \right) \begin{cases} < \sum_{j=k+1}^{\infty} \frac{1}{4j^2-1} = \frac{1}{2} \sum_{j=k+1}^{\infty} \left(\frac{1}{2j-1} - \frac{1}{2j+1} \right) = \frac{1}{4k+2} \\ > \sum_{j=k+1}^{\infty} \frac{1}{4j^2} > \frac{5}{4} \sum_{j=k+1}^{\infty} \left(\frac{1}{5j-2} - \frac{1}{5j+3} \right) = \frac{5}{20k+12} \end{cases}$$

and finally

$$I_k \begin{cases} < \exp \frac{1}{4k+2} = \frac{1}{\exp\left(-\frac{1}{4k+2}\right)} < \frac{1}{1-\frac{1}{4k+2}} = 1 + \frac{1}{4k+1}, \\ > \exp \frac{5}{20k+12} > 1 + \frac{5}{20k+12} + \frac{1}{2} \left(\frac{5}{20k+12} \right)^2 > 1 + \frac{1}{4k+2}. \end{cases} \tag{6}$$

Thus e.g. $1.0098 < I_{25} < 1.01$.

Note. Equality (3) can be rewritten as $I_k = \pi(k+\frac{1}{2})((2k)!)^2(2^k k!)^{-4}$; consequently, the estimates (6) can be also derived by brute force (with much more calculation, though) from the Stirling formula, taken for instance in the form

$$n! = \sqrt{2\pi n} n^n e^{-n} \alpha_n , \quad \frac{12n+1}{12n} < \alpha_n < \frac{12n}{12n-1} .$$

Also solved by H.L. ABBOTT, University of Alberta; M. FALKOWITZ, Hamilton, Ontario; C. FESTRAETS-HAMOIR, Brussels, Belgium; GEORGE P. HENDERSON, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; TOM LEINSTER, Lancing College, England; JOHN LINDSEY, Northern Illinois University, Dekalb; BEATRIZ MARGOLIS, Paris, France; VEDULA N. MURTY, Penn State University at Harrisburg; P. PENNING, Delft, The Netherlands; COS PISCHETTOLA, Framingham, Massachusetts; ROBERT E. SHAFER, Berkeley, California; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Breda, The Netherlands; and KENNETH S. WILLIAMS, Carleton University.

Many of the solvers gave sharper bounds than the problem asked for. Kuczma's appear to be about the best with not a great deal of calculation. Henderson notes that k need not be an integer.

The problem is an old one and, as Kuczma and others point out, comes from the usual proof of the Wallis product. Falkowitz found the given inequality, with solution, in R. Courant, Differential and Integral Calculus Vol I, *pp. 223-224. Leinster spotted it in Spivak's* Calculus, *page 328, problem 26.*

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1531. [1990: 108] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*
Prove that

$$\frac{v+w}{u}\cdot\frac{bc}{s-a}+\frac{w+u}{v}\cdot\frac{ca}{s-b}+\frac{u+v}{w}\cdot\frac{ab}{s-c}\geq 4(a+b+c),$$

where a,b,c,s are the sides and semiperimeter of a triangle, and u,v,w are positive real numbers. (Compare with *Crux* 1212 [1988: 115].)

I. *Solution by Niels Bejlegaard, Stavanger, Norway.*

By the A.M.-G.M. inequality the left hand side of the given inequality is greater than or equal to

$$\begin{aligned} 3\sqrt[3]{\prod\left(\frac{u+v}{w}\right)\cdot\frac{a^2b^2c^2}{(s-a)(s-b)(s-c)}} &= 3\sqrt[3]{\prod\left(\frac{u+v}{w}\right)\cdot\frac{16R^2r^2s^2}{r^2s}} \\ &= 3\sqrt[3]{\prod\left(\frac{u+v}{w}\right)\cdot 16R^2s}\ , \end{aligned}$$

where R is the circumradius and r the inradius, and the products are cyclic over u,v,w. So if I can show that

$$3\sqrt[3]{\prod\left(\frac{u+v}{w}\right)\cdot 16R^2s}\geq 8s\ ,$$

the problem is solved. Cubing both sides gives

$$\prod\left(\frac{u+v}{w}\right)\geq\frac{32}{27}\left(\frac{s}{R}\right)^2=\frac{32}{27}\left(\sum\sin A\right)^2\ , \tag{1}$$

where the sum is cyclic over the angles A,B,C of the triangle. Now it is known that

$$\sum\sin A\leq\frac{3\sqrt{3}}{2}\ ,$$

and also

$$\begin{aligned} \prod\left(\frac{u+v}{w}\right) &= \left(\frac{u}{w}+\frac{v}{w}\right)\left(\frac{v}{u}+\frac{w}{u}\right)\left(\frac{w}{v}+\frac{u}{v}\right) \\ &= 2+\left(\frac{u}{w}+\frac{w}{u}\right)+\left(\frac{v}{u}+\frac{u}{v}\right)+\left(\frac{w}{v}+\frac{v}{w}\right) \\ &\geq 2+3\cdot 2=8\ . \end{aligned}$$

Therefore
$$\frac{32}{27}\left(\sum \sin A\right)^2 \le \frac{32}{27}\left(\frac{3\sqrt{3}}{2}\right)^2 = 8 \le \prod\left(\frac{u+v}{w}\right) ,$$
which is (1). Equality holds when $a=b=c$ and $u=v=w$.

II. *Generalization by Murray S. Klamkin, University of Alberta.*

First we prove that
$$\sum \frac{v+w}{u}(bc)^{2p} \ge 6\left(\frac{4F}{\sqrt{3}}\right)^{2p} , \qquad (2)$$
where F is the area of the triangle, the sums here and subsequently are cyclic over u,v,w and a,b,c, and for now $p \ge 1$. Regrouping the left side of (2) and applying the A.M.-G.M. inequality to each resulting summand, we get
$$\sum \frac{v+w}{u}(bc)^{2p} = \sum\left(\frac{v}{u}(bc)^{2p} + \frac{u}{v}(ca)^{2p}\right) \ge 2(abc)^p(a^p+b^p+c^p) .$$
We now use $abc = 4RF$ (R the circumradius) and the following known inequalities:
$$a^p+b^p+c^p \ge \frac{(a+b+c)^p}{3^{p-1}},\ p\ge 1 \qquad \text{(by the power mean inequality);}$$
$$R^2 \ge \frac{4F}{3\sqrt{3}} \qquad \text{(the largest triangle inscribed in a circle is equilateral);}$$
$$(a+b+c)^2 \ge 12F\sqrt{3} \qquad \text{(the largest triangle with given perimeter is equilateral).}$$
Stringing these together, we get
$$\begin{aligned}\sum\left(\frac{v+w}{u}\right)(bc)^{2p} &\ge \frac{2(4RF)^p(a+b+c)^p}{3^{p-1}}\\ &\ge \frac{6(4F)^p}{3^p}\left(\frac{4F}{3\sqrt{3}}\right)^{p/2}(12F\sqrt{3})^{p/2} = 6\left(\frac{4F}{\sqrt{3}}\right)^{2p} ,\end{aligned}$$
i.e., (2).

We now extend the range of (2) by showing that it is also valid for $0 \le p < 1$. The rest of the proof is similar to Janous's solution of *Crux* 1212 [1988: 115-116] and uses results mentioned there. For $0 \le p < 1$, a^p, b^p, c^p are the sides of a triangle of area $F_p \ge F^p(\sqrt{3}/4)^{1-p}$. From this and the case $p=1$ of (2), we get
$$\sum \frac{v+w}{u}(bc)^{2p} \ge 32F_p^2 \ge 6\left(\frac{4F}{\sqrt{3}}\right)^{2p} .$$

Now if a,b,c are the sides of a triangle, then so are
$$\sqrt{a(s-a)}\ , \quad \sqrt{b(s-b)}\ , \quad \sqrt{c(s-c)}\ ,$$

and the area of this triangle is $F/2$. Hence from (2),

$$\sum \frac{v+w}{u}(bc(s-b)(s-c))^p \ge 6\left(\frac{2F}{\sqrt{3}}\right)^{2p} .$$

Dividing by $F^{2p} = (s(s-a)(s-b)(s-c))^p$, we obtain

$$\sum \frac{v+w}{u}\left(\frac{bc}{s-a}\right)^p \ge 6\left(\frac{4s}{3}\right)^p . \tag{3}$$

The proposed inequality corresponds to the special case $p = 1$.

As a companion inequality, we obtain

$$\sum \frac{v+w}{u} a^{2p} \ge 6\left(\frac{4F}{\sqrt{3}}\right)^p \tag{4}$$

for $p \ge 0$. We get as before (via regrouping and the A.M.-G.M. inequality) that

$$\sum \frac{v+w}{u} a^{2p} = \sum \left(\frac{v}{u}a^{2p} + \frac{u}{v}b^{2p}\right) \ge 2\sum (ab)^p .$$

For $p \ge 1$, the rest follows from the known inequalities

$$\sum (ab)^p \ge 3\left(\frac{\sum ab}{3}\right)^p , \; p \ge 1 \quad \text{(power mean)}$$

and

$$\sum ab \ge 4F\sqrt{3} .$$

The extension of (4) to the range $0 \le p < 1$ is carried out the same way as (2) was extended.

By letting $a = \sqrt{a(s-a)}$, etc. as before, we obtain a dual inequality to (4), i.e.,

$$\sum \frac{v+w}{u} a^p (s-a)^p \ge 6\left(\frac{2F}{\sqrt{3}}\right)^p . \tag{5}$$

Finally, since we always have the representation

$$a = y+z , \; b = z+x , \; c = x+y , \; s = x+y+z , \; F^2 = xyz(x+y+z) ,$$

(2)—(5) take the forms

$$\sum \frac{v+w}{u}(x+y)^{2p}(x+z)^{2p} \ge 6\left(\frac{16xyz(x+y+z)}{3}\right)^p ,$$

$$\sum \frac{v+w}{u}\frac{(x+y)^p(x+z)^p}{x^p} \ge 6\left(\frac{4(x+y+z)}{3}\right)^p ,$$

$$\sum \frac{v+w}{u}(y+z)^{4p} \ge 6\left(\frac{16xyz(x+y+z)}{3}\right)^p ,$$

$$\sum \frac{v+w}{u}(y+z)^{2p}x^{2p} \ge 6\left(\frac{4xyz(x+y+z)}{3}\right)^p ,$$

respectively, for arbitrary nonnegative numbers x, y, z, u, v, w (we have doubled p in the last two inequalities). Numerous special cases can now be obtained.

III. *Generalization by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We start by proving the following

Lemma. Let $r_1, \ldots, r_n > 0$ and put $R = r_1 + \cdots + r_n$. Then for all nonnegative $x_1, \ldots, x_n$,

$$\sum_{i=1}^{n} \frac{R - r_i}{r_i} x_i^2 \geq 2 \sum_{i<j} x_i x_j \ .$$

Proof. Indeed,

$$\sum_{i=1}^{n} \frac{R - r_i}{r_i} x_i^2 = R \sum_{i=1}^{n} \frac{x_i^2}{r_i} - \sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i \right)^2 - \sum_{i=1}^{n} x_i^2 = 2 \sum_{i<j} x_i x_j \ ,$$

where we have used the Cauchy-Schwarz inequality. Equality holds if and only if $x_1/r_1 = \ldots = x_n/r_n$. □

Putting $n = 3$ and

$$r_1 = u \ , \ r_2 = v \ , \ r_3 = w \ , \ x_1 = bc \ , \ x_2 = ca \ , \ x_3 = ab$$

we get

$$\sum \frac{v + w}{u} (bc)^2 \geq 2 \sum caab = 2abc \sum a = 4abcs \ , \qquad (6)$$

where the sums are cyclic.

[*Editor's note.* Janous now uses

$$4abcs = 16FRs \geq 32Frs = 32F^2$$

to obtain (2) for the case $p = 1$, then mimics his proof of *Crux* 1212 (exactly as Klamkin does) to obtain inequalities (2) and (3) for $0 \leq p \leq 1$. Then he uses his lemma with $n = 3, r_1 = u, r_2 = v, r_3 = w, x_1 = a^2, x_2 = b^2, x_3 = c^2$, and

$$x^2 + y^2 + z^2 \geq xy + yz + zx$$

(which is equivalent to $(x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0$) to obtain

$$\sum \frac{v + w}{u} a^4 \geq 2 \sum b^2 c^2 \geq 2 \sum caab = 4abcs \geq 32F^2 \ , \qquad (7)$$

and also extends this to obtain inequality (4) for $0 \leq p \leq 2$ only. He points out that (4) can be considered a "dual" of the published generalization of *Crux* 1051 (inequality (2) on [1986: 252], due independently to him and Klamkin), in the same way that (3) is a "dual" of his generalization of *Crux* 1212 (see [1988: 116]) and (2) is a "dual" of *Crux* 1221 (see (3) on [1988: 116]). Janous now continues...]

Obviously all the special cases stated in the solution of *Crux* 1051 can be translated literally into their "dual" versions, e.g., (v) on [1986: 254] becomes

$$\sum \frac{2s+a}{2s-a} a^4 \geq 32F^2 \ .$$

Using the stronger inequality in (7), putting $u = a^3$, etc. and dividing by $2F = ah_a = bh_b = ch_c$, we get

$$\sum \frac{b^3+c^3}{h_a} \geq \frac{2abcs}{F} = 8Rs \ .$$

Etc., etc., etc.

Applying the transformation $a \to \sqrt{a(s-a)}$, etc. to (6), we get

$$\sum \frac{v+w}{u} bc(s-b)(s-c) \geq 2\sqrt{\prod a(s-a)} \sum \sqrt{a(s-a)} \ ,$$

i.e.,

$$\sum \frac{v+w}{u} \frac{bc}{s-a} \geq 2\sqrt{\frac{abc}{\prod(s-a)}} \sum \sqrt{a(s-a)} = 4\sqrt{\frac{R}{r}} \sum \sqrt{a(s-a)} \ .$$

It seems that the right-hand quantity is greater than or equal to $8s$ (which if true would strengthen the proposed problem), but I can't prove or disprove this. Therefore I leave to the readers the following

Problem. Prove or disprove that

$$\sum \sqrt{\frac{a}{r_a}} \geq 2\sqrt{\frac{s}{R}} \ , \qquad (8)$$

where r_a, r_b, r_c are the exradii of the triangle.

Since $s - a = rs/r_a$, etc., this inequality is equivalent to the one I can't prove. Furthermore, (8) should be compared to item 5.47, p. 59 of Bottema et al, *Geometric Inequalities*, namely

$$\sum \sqrt{\frac{a}{r_a}} \leq \frac{3}{2}\sqrt{\frac{s}{r}} \ !$$

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; STEPHEN D. HNIDEI, student, University of British Columbia; MARCIN E. KUCZMA, Warszawa, Poland; D.M. MILOŠEVIĆ, Pranjani, Yugoslavia; and the proposer.

Milošević also proved inequality (7)*, with right-hand side* $4abcs$*.*

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