Some Problems in Number Theory.

§1.0 Divisibility in integers; primes and composites.

Notation. For a positive integer n (written in the decimal notation), the sum of digits of n is denoted by $s(n)$.

P1.001 Let n be a natural number satisfying the condition s(n) = s(3n). Prove that 9 divides n. Show by an example that the converse does not hold.

Hints and comments. We have $3|3n \Rightarrow 3|s(3n) \Rightarrow 3|n \Rightarrow 9|n \Rightarrow 9|n$.

P1.002 Consider all the six-digit numbers formed by using the digits 1,2,3,4,5,6 exactly once. What is the g.c.d. of all these numbers?(CRMO-1995)

Hints and comments. Let S denote the set of numbers under consideration, and let d denote their g.c.d. Then for each $a \in S$ we have $s(a) = 21$. Thus $3|d$. next, the numbers 123456 and 123465 both belong to S. Hence d divides their difference, namely 9. So the only possibilities for d are 3 and 9. However, since $s(123456) = 21$ we cannot have $d = 9$. Therefore, $d = 3$ necessarily.

P1.003 Consider the 60-digit number A formed by writing the numbers 100 to 119 consecutively; thus,

$$A := 100101102\dots119$$

Determine the highest power of 3 that divides A.

Hints and comments. Thus you are asked to find out whether

1. 3 does not divide A at all; in this case the highest
power of 3 that divides A is $1 = 3^0$;

2. 3 divides A but $9(= 3^2)$ does not divide A;
in this case the answer is $3 = 3^1$;

3. 9 divides A; inthis case the answer is 3^t for some positive integer $t \geq 2$.

P1.004 Among Mr.Smith's papers the following bill for a sum of five figures was found:

Item	Amount
6 dozen mangoes	Rs. -38. 2- p.

As seen above the first and last digits in the
amount column were faded. Assuming that each mango costs an integral
number of paise determine the missing digits.

Hints and comments. Suppose that the missing digits were x and y so that the amount to be paid is Rs $\overline{x}38$ and paise $\overline{2}y$. (Here $\overline{a}b$ means . . . and .) Thus the number of paise to be paid is $\overline{x}382y = m$ (say) paise. Since this is the price of $72 = 8 \times 9$ mangoes, the number formed by the last three digits of this number, namely $\overline{8}2y$ has to be divisible by 8.Now the only number of the form $\overline{8}2y$ which is divisible by 8 is 824. Hence $y = 4$ necessarily. This means $m = \overline{x}3824$ has to be divisible by 72. it follows that $9|x3824$ and, therefore, $9|s(x3824) = x + 17$ As x is a digit, it is an integer satisfying $0 \leq x \leq 9$. It follows that $x = 1$, necessarily. Hencethe amount ot be paid for 72 mangoes is 13824 paise, implying that each mango costs Rupee 1 and 92 paise.

P1.005 Let A be a natural number such that 99 divides A.

Prove that $s(A) \geq 18$.

Hints and comments.

We record the following remarks suggested by the divisibility test by 11:

1. For the multiples $99 \times 1, 99 \times 2, \ldots, 99 \times 100$ of 99 the sum of digits is 18. For the next multiple $99 \times 101 = 9999$ the sum of digits 'jumps' to 36, while $s(99 \times 102) = 18$, once again.

2. The equations $s_e(A) - s_o(A) = 11$ and $s_e(A) + s_o(A) = 27$ hold if and only if $s_e(A) = 19$ and $s_o(A) = 8$. Examples of positive integers in the range $10000 - 12000$ satisfying these conditions are the numbers 10989, 11979).

3. With some additional conditions too, numbers satisfying the conditions in 2 can be constructed. For example, a number A, satisfying the conditions in 2 above (so that, in particular A is divisible by 99) and has 1 and 0 as the only digits is the number 10101010101010101010101111111111111111 (10 repeated 11 times and 11 repeated 8 times).

P1.006. Suppose that n and A are positive integers.

i) Prove that if A has 3^n identical digits d, then 3^n divides A.

ii) Suppose, further, that $3 \mid d$ (i.e.,$d = 3, 6$ or9).

Prove that 3^{n+1} divides A.

iii) Suppose, finally, that $d = 9$.

Prove that 3^{n+2} divides A.

Hints and comments. P1.007. Describe the set

$$\{s(n^2) \mid n \in \mathbf{N}\}$$

completely.

P1.008. Let $A = 4444^{4444}$, $B = s(A), C = s(B), D = s(C)$.

Find D. (An IMO problem.)

Hints and comments. P1.009. Let $A = 6543^{4321}$, $B = s(A), C = s(B), D = s(C)$.

Find D. (Modification of P1.008)

Hints and comments. P1.010. For a positive integer n, let $f_1(n) := (s(n))^2$.

Thus f_1 is a map from $\mathbf{N}$ to itself.

Let $f_2 = f_1 o f_1, f_3 = f_2 o f_1$, and, recursively,

$f_{n+1} = f_n o f_1$, for each positive integer n.

Determine $f_{1990}(2^{1990})$.

Hints and comments.

P1.011. Suppose that $A = a_1a_2\dots a_{3n}$ is a $3n$-digit
number, and $B = a_ta_{t+1}\dots a_1a_2\dots a_{t-1}$ is
a cyclic permutation of A. Prove that 27 divides A if and only
if 27 divides B.

Hints and comments. 1. Certainly it is not the case that either of the conditions $27|n$ and $27|s(n)$ implies the other. Indeed, we have for $m = 27$ while $27|m$ we do not have $27|s(m) = 9$. In the reverse direction, for $m = 9981$ while $27|s(m)$ we do not have $27|m$. Hence the fact that $s(A) = s(B)$ cannot be used in conjunction with a 'divisibility test for 27' analogous to that recorded in

2. Indeed while $27|27$ it is not the case that 27 divides the cyclic permutation 72 of 27. Further, while $27|270$, a three-digit positive integer, it is not the case that $27|720$, a non-cyclic permutation of270. Hence neither of the conditions $3n$-digit integer and cyclic permutation can be omitted from the hypothesis.

P1.012.Prove the existence of a positive integer n satisfying the
conditions:

a) 1996 divides n;

b) $s(n) = 1996$.

Hints and comments. We use the facts: $s(1996) = 25, s(2 \times 1996) = s(3992) = 23$ and $1996 = 76 \times 25 + 2 \times 23$.

P1.013.Using the fact that $2^{2^k} - 1$ is a factor of $2^{2^l} - 1$
if $k \leq l$, prove that

$$2^{2^m}+1 \qquad \text{and} \qquad 2^{2^n}+1$$

are relatively prime, if m, n are positive integers such that $m \neq n$.

Hints and comments.

P1.014.Determine the set of all natural numbers n such that

ach n digit positive integer having $n-1$ digits 1 and the remaining digit as 7 is prime.

Hints and comments.

(Clearly 1,2 belong to this set while 3 doesn't. What about larger integers?)

P1.015 Show how 167 buffaloes can be distributed among four brothers in such a way that (a) no two brothers get an equal number, (b) each brother gets an integral multiple of every younger brother.

Hints and comments.

P1.016 Let n be a natural number satisfying $n \geq 3$.

Prove that both numbers

$2^n - 1$ and $2^n + 1$

cannot be prime.

Hints and comments.

P1.017 Find one three-digit prime divisor of the number

$3^{180} + 5^{135}$

Hints and comments. P1.018 Determine all primes p satisfying the condition that the numbers $p+2, p+6, p+8, p+12$ and $p+14$ are all primes.

Hints and comments.

P1.019 Determine all primes p satisfying the condition that the numbers p^2+4 and p^2+6 are also prime.

Hints and comments.

P1.020 Factorise the following numbers:8989, 999973, 99899.

Hints and comments.

P1.021 If the numbers p and $8p-1$ are both prime, then the number $8p+1$ is necessarily a composite number. Prove this.

Hints and comments.

P1.022 If the numbers p and $8p^2+1$ are both prime, then the number $8p^2-1$ is necessarily a prime number. Prove this.

P1.023 Let P_0 be a subset of the set of all primes satisfying
the condition

$$a, b \in P_0 \Longrightarrow ab + 4 \in P_0.$$

Prove that $P_0 = \emptyset$

P1.024 If p_1 and p_2 are consecutive prime numbers
(with $3 \leq p_1 < p_2$) prove that $p_1 + p_2$
has at least three factors in its complete factorization
into primes.If, further $p_1 \geq 11$ and $p_2 = p_1 + 2$
(so that p_1 and p_2 are actually twin primes), then $p_1 + p_2$
has at least four factors in its complete factorization.

P1.025 Prove that among any 18 consecutive three-digit numbers there is at least one number which is divisible by the sum of its digits.

P1.026 Four persons, A, B, C, D fix a positive integer X. Each makes three statements about the number X; out of these three, at least one is false and at least one is true. These assertions are:

A: 1. 4 divides X;

2. 9 divides X;

3. 11X is smaller than 1000.

B: 1. 10 divides X;

2. X is bigger than 100;

3. 12X is bigger than 1000.

C: 1. X is a prime;

2. 7 divides X;

3. X is smaller than 20.

D: 1. 7 does not divide X;

2. X is smaller than 12;

3. 5X is smaller than 70.

Determine the number X.

(In the question 11X, 12X and 5X mean the numbers 11 times X etc.)

P1.027 Prove that there is no six-digit number of the form
KAMALA (with K, M, L and A distinct digits with K $\neq$ 0)
which is divisible by 1887.
(A number of this form divisible by 1867 exists.)

P1.028 Determine the largest 3-digit prime factor of
the integer ${}^{2000}C_{1000}$.

P1.029 Let n be a positive integer ≥ 2. If 5 does not divide any of the integers $n-1, n$ and $n+1$, show that 5 divides n^2+1.

P1.030 Consider the six-digit integer $abcabc$ written in the usual
decimal notation. Prove that 13 divides this integer.

P1.031 Given any positive integer n, show that there exist
distinct positive integers x and y
such that $x+j$ divides $y+j$ for $j = 1, 2, 3 \ldots n$.

P1.032 If for some positive integers x and y, $x+j$ divides $y+j$ for all positive integers j, prove that $x = y$.

P1.033 Do there exist 100 positive integers such that their sum is eq ual to their least common multiple?

Hints and comments. Similar questions can be asked for numbers o ther than 100. We define a finite set of (distinct) positive integers to be an *LS-set* if each of the integers in the set divides the L.C.M. of them all.The set {1,2,3} is a 3-element LS-set, {1,4,5, 10} is a 4-element LS-set, while {1, 2, 4, 7, 14 } is

a 5-element LS-set; for each positive integer $n \geq 6$, there is

a general method of constructing LS-sets of cardinality n.)

P1.034 A finite set of (distinct) positive integers is called a "DS-set" if each of the integers in the set divides the sum of them all.Prove that every finite set of positive integers is a subset of some DS-set.

Hints and comments

1.Clearly every LS-set as defined in P1.034C is a DS-set. However, the converse is not true: the set {1, 2, 3, 6} is a DS-set which is not an LS -set.

2.Is it the case that every finite set is a subset of some LS-set? Without loss of generality, we can assume the given finite set to be the set {1,2,3, ...,n} of the first n positive integers.

3.Since {1, 2, 3, 4, 6, 8} is an LS-set, the answer to the question in 2 is 'Yes' for $n \leq 4$.

4.When $n = 5$?

Complete solution of 1.034: Let $X := \{a_1, a_2, \dots, a_s\}$ be the given s-elementic finite set. Suppose $l = lcm(X)$. We write $l = 2^{\alpha} p_1^{\alpha_1} \dots p_j^{\alpha_j}$ for some nonnegative integer j. Here, $p_i (1 \leq i \leq j)$ are all the odd primes dividing l; we have $\alpha \geq 0$ while $\alpha_i \geq 1$ for each i.

Case 1: If X is a singleton set, then X is itself a DS-set.

We may,therefore, assume that X has at least two elements.

Case 2: Assume that l is a nonnegative power of 2 so that each element of X is a power of two; thus, $X = \{2^{b_i} | 1 \leq i \leq s\}$ for some nonnegative integers b_i. Assume b_s is the largest among the b_i. Write $\sigma = \sigma(X)$, and consider the set $X_0 := X \cup \{2^j \sigma | 1 \leq j \leq b_s - 1\}$. We clearly have

$$\sigma(X_0) = 2^{b_s} \sigma$$

. Thus, X_0 is a DS-set containing X.

Case 3 ('General' case) : If l is not a nonnegative power of 2 then, in the above notation, the integer j is positive. using binary epansion we write

$$p_1^{\alpha_1} = 1 + 2^{\beta_{11}} + \cdots + 2^{\beta_{1 l_1}}$$

for some positive integer l_1.

Consider the set

$$X_1 = X \cup \{2^{\beta 1 i} | 1 \leq i \leq l_1\}$$

Then

$$\sigma(X_1)$$

P1.035 Prove that for each natural number n, the number

$$A_n = \frac{1}{3}(2^{2^{n+1}} + 2^{2^n} + 1)$$

is an integer divisible by 7.

In particular, A_n is composite for each $n \geq 2$.

P1.036 Prove that for each natural number n there exists a set of n natural numbers satisfying the condition “the difference of any two members of the set divides the larger of the two”.

Hints and comments.

P1.037 Determine all six-digit natural numbers $a_1a_2a_3a_4a_5a_6$ satisfying the conditions

*(i)*the digits $1, 2, 3, 4, 5, 6$ are used exactly once.

(in set theoretic notation

$$\{a_1, a_2, a_3, a_4, a_5, a_6\} = \{1, 2, 3, 4, 5, 6\})$$

and

(ii) for each $i = 1, 2, 3, 4, 5, 6$, the integer i divides the i-digit number $a_1a_2a_3 \ldots a_i$.

P1.038 Show that if n is a natural number such that $4^n + 2^n + 1$ is a prime, prove that n must be a power of 3.

P1.039 Show that given an integer $n \geq 12$ there exist composite numbers x and y such that $n = x + y$.

P1.040 Show that given an even integer $m \geq 40$ there exist two odd composite numbers x and $y$such that $m = x + y$.Further, 40 is the smallest element in the set $W := \{w|w$ is an even positive integer with the property that for each even integer v satisfying $v \geq w$ there exist two odd composite numbers x and y such that $m = x + y$.

P1.041 Can there exist five distinct positive integers such that the sum of every three of them is a prime? Answer with justification.

Hints and comments.The existence of infinitely many pairs of positive integers with prime sums is easily verified; namely, for each odd prime p, writing $p = 2s + 1$ for some integer s, the sum of the integers s and $s + 1$ equals p. Again, if $p = 2s + 1 \geq 7$, so that $s \geq 3$, we have $p = 1 + (s - 1) + (s + 1)$, and, thus, there exist infinitely many triples of postive integers with prime sums.However, we have: A. It is impossible to have three distinct (or even non-distinct - with the exception of the triples (1, 1, 1), (1, 1, 2), (1, 2, 1) and (2, 1, 1) - positive integers a, b, c such that three pairwise sums $a + b, b + c, c + a$ are all primes.

B. It is possible to have four distinct positive integers (a, b, c, d) such that the sum of every three among these four is a prime. Note that 0. For every quadruple (a, b, c, d) with $a, b, c, d \neq$, satisfying the given condition,the integers a, b, c and d must be all odd.

1. If distinctness is not required then the quadruple $(1, 1, 1, 1)$ is such that for every three among these four integers their sum ($= 1 + 1 + 1 = 3$) is a prime; clearly, this is the only quadruple of the form (a, a, a, a) satisfying

the given condition.

2. Suppose that the distinctness condition is replaced by the condition ‘ non-distinct and not all equal . Then there are three cases (upto permutations)to be considered: 2.1. The first possibility is of quadruples of the form (a, a, a, b) (with $a \neq b$. Clearly $a + a + a$ prime implies $a = 1$. Further, for each prime $p \geq 5$, the given condition is satisfied by the quadruple $1, 1, 1, p - 2$ (producing the ‘triple sums’ 3 and p).

2.2. The second possibility is of quadruples of the form (a, a, b, b).(with $a \neq b$). This produces the ‘triple sums’ $a + 2b$ and $2a + b$.Clearly, the condition $a + 2b = 3 \Rightarrow a = 1 = b$ and so if (unequal) primes $a + 2b = p$ and $2a + b = q$ (say), both satisfying $p, q \geq 5$ are to be produced we must have $p + q = 3(a + b) \geq 12$ Clearly, the integers p, q cannot leave the same remainder on dividing by 3; so assume, without loss of generality $p = 3k + 1$ and $q = 3l + 2$ (for some positive integers k and l). Then we have:

$$p = 2a + b, q = a + 2b \Leftrightarrow a = \frac{2p - q}{3} \text{andb} = \frac{\text{2q} - \text{p}}{3}$$

It follows that (to produce positive integers a, b) we must have $2p > q$ and $2q > p$.

(Explicit examples: for $a = 1$ and $b = 3$ - with the corresponding quadruple $(1, 1, 3, 3)$ - we have $p = 7$ and $q = 5$; for $a = 1$ and $b = 5$ - with the corresponding quadruple $(1, 1, 5, 5)$ - we have $p = 11$ and $q = 7$) Infinitely many pairs (a, b)? 2.3. The third, final, possibility is of quadruples of the form (a, a, b, c). This produces the sums $a + b + c, 2a + b$ and $2a + c$.

Examples: The values $a = 3, b = 1$ and $c = 7$ give rise to the prime ‘triple sums’ $a + b + c = 11, 2a + b = 7$ and $2a + c = 13$; the values $a = 3, b = 11$ and $c = 5$ give rise to the prime ‘triple sums’ $a + b + c = 19, 2a + b = 17$

and $2a + c = 11$; the values $a = 3, b = 11$ and $c = 17$ give rise to the prime 'triple sums' $a + b + c = 31, 2a + b = 17$ and $2a + c = 23$. Infinitely many triples (a, b, c)? Indeed with $a = 3$ the quadruple $(3, 3, b, c)$ produces the sums $3 + b + c, 6 + b$ and $6 + c$. Since these three integers are to be prime, neither b nor c can be divisible by 3. Further, b and c cannot leave different remainders on dividing by 3. Hence we have the cases: Case 1. $b = 3k + 2$ and $c = 3l + 2$ for some (necessarily odd, since b and c are both odd) positive integers k and l. In this case after choosing, for a given triple (p_1, p_2, p_3) of primes satisfying and

Case 2. $b = 3k + 1$ and $c = 3l + 1$ for some (necessarily even, since b and c are both odd) nonnegative integers k and l. In this case after choosing, for a given triple (p_1, p_2, p_3) of primes satisfying and

Clearly to get four distinct positive integers a, b, c, d such that the triple sums are all primes, it is advisable to look at the 'smallest' values. Since $a = 1, b = 3, c = 5$ produces a composite sum, discarding the number 5 and replacing it by 7 we get the condition ' $d + 1 + 3 = d + 4, d + 1 + 7 = d + 8$' and $d + 3 + 7 = d + 10$ are all primes. Clearly $d = 9$ satisfies this condition; so also does $d = 99$.

P1.042 Let n be a positive integer .Suppose that $p_1, \cdots, p_n$ are n prime numbers (not necessarily distinct) such that each $p \geq 5$. Assume that 6 divides $p_1^2 + \cdots + p_n^2$. Prove that 6 divides n.

P1.043 Find all primes p such that the integer $p^2 + 11$ has exactly six different divisors (including 1 and the number itself).

Notation. For a set A, the number of elements of A is denoted by $n(A)$.

P1.044 Find $Min\{n(A) \mid$ there exists a function $f : N \longrightarrow A$ such that whenever $|i-j|$ is a prime number we have $f(i) \neq f(j)\}$
(Proposed by Romania in Balkan Math Olympiad,)

P1.045 Define a sequence p_n as follows :
$p_1 = 2$ and for $n \geq 2$,
$p_n =$ the largest prime divisor of $p_1 \cdots p_{n-1} + 1$
Prove that p_n can never be 5.

P1.046 Show the existence of 7 primes $p_1 \cdots p_7$ satisfying

$$(i) p_1 < p_2 \cdots < p_7 < 1000$$

and

$$(ii) p_2 - p_1 = p_3 - p_2 = \cdots = p_7 - p_6$$

P1.047 Let x, y be distinct prime numbers. Show that $x + y$ does not divide $x^2 + y^2$.

P1.048 Find positive integers x, y, z such that $x < y < z$ and
$1/x - 1/xy - 1/xyz = 19/97$

P1.049 Let $f(Y)$ be the polynomial $Y^4 + Y^3 + Y^2 + Y + 1$. Prove that if

t is an odd positive integer then $f(5^t)$ is a composite number.

P1.050 Prove that $(5^{125}-1)/(5^{25}-1)$ is a composite number.

P1.051 Show that there exist three integers A, B, C each greater than 10^{100} such that $5^{1985}-1=ABC$

P1.052 Let a, b, c, d be positive integers such that $ab=cd$. Prove that $a+b+c+d$ is not a prime number.

P1.053 Find the remainder when 2^{1990} is divided by 1990.

P1.054 Find the remainder when 19^{92} is divided by 92.

P1.055 Find the tens digit in $6^{101}-6$.

P1.056 Find the remainder when 3^{1998} is divided by 1998.

P1.057 Prove that for each integer $n \geq 2$ the integer n^4+4 is composite.

P1.058 Prove that for each integer $n \geq 2$ the integer n^4+4^n is composite.

P1.059 Show that there exist two integers A, B each of them bigger than 10^{50} such that

$$199^4+4^{199}=AB$$

.

P1.060 Let r be the remainder obtained by dividing a prime number p by 30. Show that either $r=1$ or r itself is a prime number.

P1.061 Prove that for each non-negative integer n the tens' digit in 3^n is even.

P1.062.For each non-negative integer n, let $a_n = 20 + n^2$ and let $d_n = gcd(a_n, a_{n+1})$. Describe, with proof, the set $\{d_n | n \in \mathbb{N} \cup \{0\}\}$.

P1.063.For each non-negative integer n, let $a_n = n^2 + 1$ and let $d_n = gcd(a_n, a_{n+1})$. Describe, with proof, the set $\{d_n | n \in \mathbb{N} \cup \{0\}\}$.

P1.064.Find all triples (p, q, n), where p, q are primes, n is an even number such that $n > 2$, satisfying the equation

$$p^n + p^{n-1} + \cdots + p + 1 = q^2 + q + 1$$

.

P1.065. Determine all triples of positive integers a, b, c

satisfying the conditions:

i) $1 < a < b < c$; and

ii) $(a-1)(b-1)(c-1)$ divides $(abc-1)$.

P1.066. Show that the integer $53 \times 83 \times 109 + 40 \times 66 \times 96$ is not a prime number.

§ 1.1 Squares, cubes, perfect powers.

Terminology: we call a positive integer x a *perfect power* if there exist positive integers y and z, both greater than 1, such that $x = y^z$. We call x *a prime power* if there exists a prime p such that $x = p^t$ for some positive integer t.(Thus primes are also prime powers; a prime power which is not a prime number is a perfect power; 36 is a perfect power which is not a prime power.).

P1.101 Call a positive integer *special* if it ends with the digit 5 and its digits are non-decreasing. Call x $super - special$ if both x and x^2 are special. Prove that there are infinitely many super-special numbers. (Examples of super-special numbers are 5, 15 and 35, whose squares are 25, 225 and 1225)

P1.102 Let k be a positive integer. Show that there exist infinitely many triples $\{a, b, c\}$ of positive integers such that $ab - k, bc - k, ca - k$ are all squares.

P1.103 Show that a three or four digit positive integer of the form $abab$ can never be a square.

P1.104 Determine all positive integers satisfying the following two conditions:

i) m is a square;

ii) m is of the form $aabb$.

P1.105 Prove that $2^p + 3^p$ is not a perfect power if p is a prime number.

P1.106 Consider the number x satisfying the following conditions:

i) x has more than one digit;

ii) X has the form $aaaa\dots a$ (i.e., the digit a is repeated in x.)

Show that x cannot be a square.

P1.107 Prove that for any integer $k > 1$ and any positive integer n, the integer n^k is (can be expressed as) a sum of n consecutive odd numbers.

P1.108 Prove that in any set of 181 square integers, one can always find a subset of 19 numbers, sum of whose elements is divisible by 19.

(INMO 1994)

P1.109 Find all pairs of positive integers (x, y) satisfying the following equation:

$$1! + 2! + 3! + \cdots + x! = y^2$$

P1.110. Find all solutions of the equation

$$m! + 2 = y^2$$

P1.111. Let m = 3, 4, 5 or 6 .Prove that the sum of m consecutive squares cannot be a square.

P1.112 Find an example of eleven consecutive squares whose sum is a square.(An Israel M.O ,Q)

P1.113 Given a positive integer k, show that there exist infinitely many positive integers n such that $n2^k - 7$ is a square.(IMOTC)

P1.114 Determine the set $\{n \in I\!N \cup \{0\} | -5^4 + 5^5 + 5^n$ is a square$\}$.(Greek Problem MOAW p 11)

P1.115 Let $f(Y) = Y^4 + Y^3 + Y^2 + Y + 1$ .Show that f(n) is a square precisely for the following integer values of n: n = -1, n = 0 and n = 3.

P1.116 Show that for each positive integer n there exists an arithmetic progression

$$a_1 < a_2 < a_3 < \cdots < a_{2n+1}$$

such that the product of all the terms is a square.

P1.117 Find all non-negative integers m, n such that
$2^m 3^n + 1$ is a perfect square.

P1.118 Determine the set of integers n such that $n^2 + 19n + 92$ is a square..

P1.119 Prove that if 1 is added to the product of four consecutive integers we lways get a square.

P1.120 Prove that if x, y, u, v are consecutive integers in an arithmetic progression, with common difference d, then the integer $xyuv + d^4$ is necessarily a square.

P1.121 Let x and y be consecutive integers. Prove that $x^2 + x^2y^2 + y^2$ is a square.

P1.122 Let x be an integer. If the tens' digit in x^2 is 7, what is necessarily the units' digit? Answer with justification.

P1.123 Determine the set of all positive integers n for which $n.2^{n-1}$ is a perfect square.

P1.124 Determine the set of all positive integers n for which $n.2^{n-1}+1$ is a perfect square.

P1.125 If $2 = p_1 < p_2 < p_3 < \cdots < p_n$ where p_i are primes, show that $p_1 \cdots p_n + 1$. can never be a perfect square. (Excursion, p.7,p.21)

P1.126 Can there exist fourteen fourth powers (not necessarily distinct or nonzero) whose sum is 1599?

§1.3 **Factorials, binomial coefficients, binomial theorem.Wilson's theorem.**

P1.301 ($\equiv$ P1.401.)How many zeroes are there at the end of 1000!? How many at the end of 1000! + 10!? Answer with justification.

P1.302 Find all solutions (in non-negative integers) of the equation

$$2(n!) = m!(m! + 2)$$

P1.303 In a group of 250 persons, for each pair (A, B)
A speaks a language that B doesn't speak and B speaks a language
that A doesn't speak. What is the minimum number of languages
the people in that group could be speaking?

P1.304 Give one pair of positive integers a, b
such that

i) 7 does not divide a, 7 does not divide b;

ii) 7 does not divide $a + b$; and

iii) 7^7 divides $(a + b)^7 - a^7 - b^7$

Hints and comments.

P1.305 Show that $19^{93} - 13^{99}$ is a positive integer
divisible by 162.

P1.306 Show that there do not exist positive integers x, y, z, n
with $n \geq z$ satisfying the condition

$$x^n + y^n = z^n$$

P1.307 Show that there do not exist non-negative integers $k, m \in \mathbb{N} \cup \{0\}$

such that

$$k! + 48 = 48(k+1)^m$$

.

§1.4 **Unique factorization in integers.**

P1.401 How many zeroes are there at the end of 1000!? How many at the en d of 1000! + 10!? Answer with justification.

P1.402 For a positive integer n, define $A(n)$ to be

$\frac{(2n)!}{(n!)^2}$. Determine positive integers n

for which:

i) $A(n)$ is an even number;

ii) $A(n)$ is a multiple of 4.

P1.403 If a, b, c are positive integers such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ with $gcd(a, b, c) = 1$, prove that $a + b$ is a square.

P1.404 *an INMO-1996 problem.* Show that there do not exist positive integers m, n such that

$$\frac{m+1}{n} + \frac{n}{m} = 4$$

holds.

P1.405 *an INMO-1986 problem.*

If a, b, x, y are integers greater than 1 such that x and y have no common factors except 1 and $a^x = b^y$, show that $a = m^y$ and $b = m^x$ forsome integer m.

This is problem 5 from PPPP.

P1.406 *an AMTI problem.* Let a, b, c, d be positive integers satisfying the conditions $a^2 = b^3, c^4 = d^5$ and $b - d = 19$. Determine their values.

P1.407 For a positive integer n, let $d(n)$ be the number of distinct positive integers dividing n (including 1 and n itself). Determine the set of all positive integers n such that $d(n) = n/3$.

P1.408 Find all primes p for which the quotient $(2^{p-1} - 1)/p$ is a square.

P1.409 Let x, y be distinct real numbers.

Write

$$Q_i = Q_i(x, y) = \frac{x^i - y^i}{x - y}$$

for each positive (non - negative) integer i.Suppose that the ratio $Q_i(x, y)$ is an integer for four consecutive values of i.Show that $Q_i(x, y)$ is an integer for each integral value of i .

P1.410 Determine, with proof, all arithmetic progressions with integer terms such that for all positive integers n, the sum of the first n terms is a square.

P1.141. Show that there exist infinitely many quadruples (a, b, c, d) of integers satsfying the condition

$$a^3 + b^3 + c^3 + d^3 = 1999$$

.

§1.5 **Special problems.**

P1.501 Determine all triples (a, b, c) of positive integers such that

$$(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c})=2$$

1.1.8P.02.Determine all triples (a, b, c) of positive integers such that

$$(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c})=3$$

P1.502 Determine all triples (a, b, c) of positive integers such that

$$(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c})=4$$

P1.503 Determine all triples (a, b, c) of positive integers such that

$$(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c})=5$$

P1.504 Determine all triples (a, b, c) of positive integers such that

$$(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c})=6$$

P1.505 Determine all triples (a, b, c) of positive integers such that

$$(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c})=7.$$

P1.506 Show that the equation

$$x^2 + y^5 = z^3$$

has infinitely many solutions in integers x, y, z for which

$xyz \neq 0$

P1.507 Examine the question of the number of solutions of the equation

$$x_1^3 + x_2^5 + x_3^7 + x_4^{11} = x_5^{13}$$

in 5-tuples $(x_1, x_2, x_3, x_4, x_5)$ of positive integers.

(More specifically, determine whether this number is zero, nonzero but finite or infinite.)

P1.508 Determine all triples (x, y, z) of positive integers such that

$$x^{y^z}.y^{z^x}.z^{x^y} = 1990^{1990}xyz.$$

(Australian-Polish Mathematical Competition [APMC] 1990))

P1.509 Determine all pairs (x, y) of positive integers such that

$$x^y.y^x = 10^{10}.$$

P1.510 Let n be a positive integer. Prove that the equation

$$x^2 + y^2 = z^n$$

always has a solution (x, y, z) in natural numbers.

P1.511 Let $f(n)$ denote the number of permutations

$(a_1, a_2, a_3, \ldots a_n)$

of the numbers $(1, 2, 3, \ldots, n)$ satisfying the conditions

$$i) a_1 = 1; and$$

$$ii) \mid a_i - a_{i+1} \mid \leq 2 \qquad \forall i = 1, 2 \ldots, 1995$$

Determine whether or not 3 divides $f(1996)$

P1.512 The integers $1, 2, 3, \ldots 1993$ are arranged in an order such that

each value is either strictly bigger than all the preceding values

or is strictly smaller than all the preceding values.

How many such arrangements are there?

P1.513 Call a permutation π on $\{1, 2, \ldots, n\}$

good if and only if $\mid \pi(j) - j \mid$ is constant

for all j satisfying $1 \leq j \leq n$.

Determine the number of good permutations for $n = 1996$.

P1.514 If a and b are positive integers such that 7 divides

$a^2 + b^2$, prove that 49 divides $a^2 + b^2$.

P1.515 Show that the equation,

$$x^2 + y^2 + z^2 = (x - y)(y - z)(z - x) \qquad (*)$$

has infinitely many solutions (x, y, z) in integers.

P1.516 Show that there does not exist a seven digit number A having all its digits distinct such that 5 divides the sum of each three consecutive digits of A. Show by examples that six-digit numbers satisfying this condition and beginning with any given digit do exist.

P1.517 Let a, b be positive integers.Define a sequence x_i
for $i \geq 0$ as follows: $x_0 = 1$ and recursively,

$$x_i = ax_{i-1} + b$$

for each $i \geq 1$
Prove that there are infinitely many composite numbers
in this sequence.

P1.518 Consider the sequence

$$x_1 = 123$$

$$x_2 = 123123$$

$$x_3 = 123123123$$

.... If p is a prime different from 2 or 5, prove that this sequence contains infinitely many terms divisible by p.

Notation . For a positive integer n, written in decimal notation, the last digit of n is denoted by $ld(n)$.

P1.519 Suppose the last digit of the positive integer a_1 is different from 5 or 0.(I.e.,$ld(a_1) \neq 5, \neq 0$) For $n \geq 2$ define a_n inductively by

$$a_n = a_{n-1} + ld(a_{n-1})$$

Prove that the sequence $a_1, a_2, ...a_n$ contains infinitely many powers of 2.

P1.520 The sequence (x_n) is given by

$$x_n = \frac{1}{4}\left((2+\sqrt{3})^{2n-1} + (2-\sqrt{3})^{2n-1}\right)$$

, for $n \in N$. Prove that each x_n is equal to the sum of squares of two consecutive integers.

P1.521 For a natural number $k \geq 1$ let $p(k)$ denote the least prime which is not a divisor of k. If $p(k) \geq 2$, define $q(k)$ to be the product of all primes less than $p(k)$, and if $p(k) = 2$, set $q(k) = 1$. Consider the sequence

$$x_0 = 1, x_{n+1} = \frac{x_n p(x_n)}{q(x_n)}$$

for $n = 0, 1, 2, \ldots$

Determine all natural numbers n such that $x_n = 111111$.

P1.522 The increasing sequence 1,3,4,9,10,12,13,... consists of positive integers which are sums of (one or more) distinct powers of 3. What is its 100th term?

P1.523. Let a, b, c, d be positive integers such that $ad - bc > 1$. Prove that at least one of the numbers a, b, c, d is not divisible by $ad - bc$.

P1.524. Prove that there exist $a, b, c, d \in N$ such that for each $y \in N\{a, b\}$

there exists $x \in N$ such that

$$\sqrt{56} < x/y < \sqrt{58}$$

. P1.525 The cube of an n-digit number has m digits. Can we have $m + n = 2001$?

P1.526.If x, y, z satisfy $x^2 + y^2 + z^2 = 1993$, show that $x + y + z$ cannot be a square.

P1.527.If a, b, c are positive integers satisfying the condition

$$ab/c + bc/a + ca/b = 3 \qquad (*)$$

prove that $a = 1, b = 1$ and $c = 1$.

§1.6 More problems.

P1.601.Let p be a prime number satisfying $p > 30$.Show that on dividing p by 30 the remainder is either 1 or a prime (lying between 1 and 29, necessarily.

Hints and comments. Similar assertions can be made on replacing 30 by the following integers: $2, 3, 4, 6$ (since the remaiders are (respectively) the sets $\{1\}, \{1, 2\}, \{1, 3\}$ and$\{1, 5\}$).

Q. Is there any natural number, apart from $2, 3, 4, 6$ and 30, for which a similar assertion can be made?

P1.602.Determine all pairs (x, y, p) of (positive) primes which satisfy $x^2y = p^2 + 11$.