

On some simple Geometrical applications

5th September, 2008

Manjil P. Saikia

manjil.saikia@gmail.com

"Inspiration is needed in geometry, just as much as in poetry."

-- Aleksandr Pushkin

Abstract: This article is based on a lecture given at Kaliabor College Kaliabor(India) for the mathematical Olympiad enthusiasts. This article discusses some simple geometrical properties with some applications.

1. Warm Up Problem: On the sides BC, CA, AB of triangle points D, E and F are taken in such a way so that $\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = 2$. Show that the area of the triangle formed by the lines AD, BE, CF is one seventh the area of the triangle ABC .

2. Some important results:

2.1(Ceva) If the points D, E, F are taken on the sides BC, CA, AB of triangle ABC such that the lines AD, BE, CF are concurrent at a point O , then $\frac{BD}{DC} \frac{CE}{EA} \frac{AF}{FB} = 1$.

2.2 (Menelaus) If a transversal cuts the sides BC, CA, AB (suitably extended) of triangle ABC in points D, E, F respectively then $\frac{BD}{DC} \frac{CE}{EA} \frac{AF}{FB} = -1$

2.3 (Stewart) If point D divides the base BC of triangle ABC in the ratio $\frac{BD}{DC} = \frac{m}{n}$ then we have $mAB^2 + nAC^2 = mBD^2 + nCD^2 + (m+n)AD^2$.

2.4 (Ptolemy) The rectangle obtained by the diagonals of a cyclic quadrilateral is equal to the sum of the rectangles contained by the pairs of its opposite sides.

2.5 (Euler) In any triangle ABC , $OI^2 = R^2 - 2Rr$ where O, I are the centres and R, r are the radii, respectively of the circumcircle and incircle of triangle ABC .

2.6 (Apollonius) If AD is a median of the triangle ABC , where D is a point on BC then $AB^2 + AC^2 = 2(BD^2 + AD^2)$

3. Applications: Solve the following problems by applying the theorems mentioned above or otherwise.

3.1 Prove that the altitudes of a triangle are concurrent.

3.2 Points E, F on the sides CA, AB of triangle ABC such that FE is parallel to BC ; BE, CF intersect at X . Prove that AX is a median of triangle ABC .

3.3 The external bisector of angle A of triangle ABC meets BC produced at L and the internal bisector of angle B meets CA at M . If LM meets AB at R then prove that CR bisects the angle C .

3.4 (Cosine rule) In triangle ABC , AD is perpendicular to BC , prove that $AB^2 = BC^2 + CA^2 + 2BC \cdot DC$ when angle C is obtuse.

3.5 Consider an isosceles triangle. Let r be the radius of its circumcircle and p the radius of the inscribed circle. Prove that the distance between the centres of these circles is $d = [r(r - 2p)]^{0.5}$.

3.6 (Brahmagupta) If in triangle ABC , AD is the altitude and AE is the diameter of the circumcircle through A , then prove that $AB \cdot AC = AD \cdot AE$.

3.7 In triangle ABC , prove that $m_a + m_b + m_c < a + b + c < \frac{4}{3}(m_a + m_b + m_c)$, where a, b, c are the sides of the triangle and m_a, m_b, m_c are the medians of the triangle.

3.8 In triangle ABC , G is the centroid, then prove that $AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$.

3.9 The points A_0, A_1, A_2, A_3, A_4 divide a unit circle into five equal parts. Prove that the chords A_0A_1, A_0A_2 satisfy $(A_0A_1 \cdot A_0A_2)^2 = 5$.

3.10 Prove 2.6.

4. Geometrical Inequalities:

4.1 (Andreescu-Enescu Lemma/Engel's Lemma) If $a_i (i = 1, 2, \dots, n)$ and $b_i (i = 1, 2, \dots, n)$ are real numbers with $b_i (i = 1, 2, \dots, n)$ are not equal to 0, then we have, $\sum \frac{a_i^2}{b_i} \geq \frac{(\sum a_i)^2}{\sum b_i}$, ($i = 1, 2, 3, \dots, n$) where equality occurs when $a_i = kb_i, k \in \mathbb{Z} (i = 1, 2, \dots, n)$.

4.2 (Nesbitt) If a, b, c are the sides of a triangle, show that $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$ where equality occurs in case of an equilateral triangle.

4.3 (IMO 1983/6) Let a, b, c be the lengths of the sides of a triangle then prove that $a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0$.

4.4 Let the bisector of angle C of triangle ABC meet the side AB at D . Prove that $CD^2 < AC \cdot BC$.

4.5 (IMO 1995/2) Let a, b, c be positive reals such that $abc=1$, then prove that $\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$.

4.6 Give a geometrical significance of the **A.M-G.M** inequality for two variables.

5. Parting Gift:

(Erdos-Mordell) If from point P inside a triangle ABC , perpendiculars PH_1, PH_2, PH_3 are drawn to its sides then $PA + PB + PC \geq 2(PH_1 + PH_2 + PH_3)$.

Website: http://www.manjil.vndv.com/activities/olymp_geo.pdf

[P.S.: Please note that the solutions of the problems are not given in this version of the lecture!]