PROBLEMS IN ELEMENTARY NUMBER THEORY–CATEGORY III

Manjil P. Saikia[1]
Department of Mathematical Sciences,
Tezpur University, India

1. *Let* $a_1, a_2, \cdots, a_n$ *be non-negative integers, then* $(a_1 + a_2 + \ldots + a_n)!$ *is divisible by* $a_1!.a_2!. \cdots .a_n!$.

Hojoo Lee, Tom Lovering and Cosmin Pohoată

2. *Let* m *and* n *be arbitary non-negative integers. Prove that* $\frac{(2m)!(2n)!}{m!n!(m+n)!}$ *is an integer.*

IMO, 1972

3. *There are* n *windows in a classroom which are all closed and numbered* 1 *through* n*. The teacher calls* n *students and then asks the* n*-th student to either close an open window or open a closed one whose number is a multiple of* n*. After the whole exercise which windows will be left open?*

Manjil P. Saikia

4. *Does there exist an infinite set* A *of positive integers with the property that the sum of the elements of any finite subset of* A *is not a perfect power?*

Kvant, 1982

5. *The* $2n$ *move chess game has the same rules as the regular one, with only one exception, each player has to make* $2n$ *consecutive moves at a time. Prove that white (who plays first) always has a non-losing strategy.*

Manjil P. Saikia, Bangladesh Mathematical Olympiad Training Camp 2008

6. *Prove that* $\forall n \in \mathbb{Z}^+$,

$$\lfloor \frac{n+1}{2} \rfloor + \lfloor \frac{n+2}{2^2} \rfloor + \ldots + \lfloor \frac{n+2^k}{2^k+1} \rfloor + \ldots = n.$$

IMO, 1968

7. *If* x *is a positive real number and* n *is a positive integer, prove that,*

$$\lfloor nx \rfloor \geq \frac{\lfloor x \rfloor}{1} + \frac{\lfloor 2x \rfloor}{2} + \ldots + \frac{\lfloor nx \rfloor}{n}.$$

USAMO, 1981

8. *Find the sum* $1.1! + 2.2! + \ldots + n.n!$.

Canada, 1969

9. *If* $n > 1$ *then prove that* $n^4 + 4^n$ *is never a prime.*

[1]Email: manjil.saikia@gmail.com, manjil_msi09@agnee.tezu.ernet.in

Kürschak, 1978

10. *Show that the equation* $x^2 + y^2 + z^2 = 2xyz$ *has no integral solutions except* $x = y = z = 0$.

Arthur Engel

11. *If* a, b *and* $q = \frac{a^2+b^2}{ab+1}$ *are integers, then prove that* q *is a perfect square.*

IMO, 1988

12. *Solve the system of equations:*

$$x + \lfloor y \rfloor + \{z\} = 200.0$$
$$\{x\} + y + \lfloor z \rfloor = 190.1$$
$$\lfloor x \rfloor + \{y\} + z = 178.8.$$

Australia,1999

13. *If* $a \equiv b \ (mod\ n)$, *then show that* $a^n \equiv b^n \ (mod\ n^2)$.

Titu Andreescu, Dorin Andrica and Zuming Feng

14. *Find the number of positive integers* x *which satisfy the equation:*

$$\lfloor \frac{x}{99} \rfloor = \lfloor \frac{x}{101} \rfloor.$$

RMO, 2001

15. *Let* a, b *and* c *be three natural numbers such that* $a < b < c$ *and* $gcd(c-a, c-b) = 1$. *Suppose there exists an integer* d *such that* $a + d, b + d$ *and* $c + d$ *form the sides of a right angled triangle. Prove that there exists integers* l *and* m *such that* $c + d = l^2 + m^2$.

RMO, 2007

16. *Prove that the fraction* $\frac{21n+4}{14n+3}$ *is irreducible for every positive integer* n.

IMO, 1959

17. *If the distinct prime factors of* n *are* $p_1, p_2, \cdots, p_k$, *then prove that*

$$\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_k}).$$

Leonhard Euler

18. *Prove that*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}.$$

Catalan

19. *Let* m *and* n *be positive integers such that*

$$\frac{m}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that $1979 \mid m$.

IMO, 1979

20. *Let p be a prime number and m and n be two integers considered in the following way, $m = a_k p^k + a_{k-1}p^{k-1} + \ldots + a_1 p + a_0$ and $n = b_l p^l + b_{l-1}p^{l-1} + \ldots + b_1 p + b_0$, where all a_i, b_i's are non-negative integers less than p. Then prove that,*

$$\binom{m}{n} = \Pi_{i=0}^{max(k,l)} \binom{a_i}{b_i}.$$

E. Lucas

21. *If p is a prime, then prove that for any integer a, $p \mid a^p + (p-1)!a$ and $p \mid (p-1)!a^p + a$.*

David M. Burton

22. *If p is an odd prime, then prove that*

$$1^p + 2^p + \ldots + (p-1)^p \equiv 0 \ (mod\ p).$$

David M. Burton

23. *Find the last two digits of 27^{1729}.*

Manjil P. Saikia

24. *Show that there are infinitely many non-zero solutions to the equation $a^3 + b^3 + c^3 + d^3 = 0$.*

Manjil P. Saikia

25. *If A and B are two Facebook friends then AB forms a Facebook line. $ABCD$ will be a Facebook polygon if A and B, B and C, C and D and A and D are friends. We can draw diagonals joining two points iff they are Facebook friends. Find out the largest sample size of people needed to form a Facebook quadrilateral with no diagonals.*

Mohaimin Elements and Manjil P. Saikia

26. *Prove that a natural number $p > 1$ is a prime iff $\binom{n}{p} - \lfloor \frac{n}{p} \rfloor$ is divisible by p for every non-negative integer n.*

Manjil P. Saikia and Jure Vogrinc

27. *Prove that a natural number $p > 1$ is a prime iff $\binom{q}{p} - \lfloor \frac{q}{p} \rfloor$ is divisible by p for every prime number q.*

Manjil P. Saikia and Jure Vogrinc

28. *Prove that if given any natural number n, the set $\{1, 2, 3, \cdots, 2^{n+1}\}$ can be partitioned into two subsets, A_n and B_n, each of size 2^n, such that $\sum_{a \in A_n} a^k = \sum_{b \in B_n} b^k$ for $k = 1, 2, \cdots, n$.*

B. Sury

29. *Show that $7x + 9y = n$ has non-negative integer solutions $\forall n > 47$, $n \in \mathbb{N}$.*

Manjil P. Saikia

30. *Prove that the number of representations of $8n+4$ as a sum of four odd squares is twice that when it is represented as the sum of four even squares.*

Nayandeep Deka Baruah, Shawn Cooper and Micheal D. Hirshchorn